MA 1: SESSION 1

1. Introduction

Calculus is one of the greatest achievements of mankind. The main definitions and theorems (without proofs) can fit on both sides of a single sheet of notebook paper, but the techniques are powerful and the applications are almost limitless. It's hard to think of a scientific field that has not been fundamentally transformed as a result of calculus and the thinking it inspired.

However, calculus is also complicated to learn. It takes bits and pieces from many parts of mathematics: algebra, geometry, a dash of combinatorics, as well as many other subtopics that you didn't know could be studied by themselves. Despite the clarity of the main theorems of calculus, the ideas and applications derive from many different sources, and it can take some time to put them all together, or to see how you can turn the elegant abstractions of calculus into something you can see and feel.

Mathematics, as an academic discipline, is certainly more cumulative than most subjects, but there is a good reason why it feels like you need to use all the mathematics you know in order to study calculus. Essentially, ever since the Cold War, the math curriculum in the United States has been leading up to learning calculus, mainly so we could train as many scientists and engineers as possible, and historical momentum has kept it in place. Thus, the mathematics you learned in high school is probably more related to physics (e.g. calculus, analytic geometry) than something like computer science (e.g. discrete mathematics, algorithms, logic) or the social or medical sciences (e.g. statistics, probability). (If you know somebody who was in elementary school in the U.S. in the post-Sputnik era, try and ask if they had to learn "New Math," which started with set theory.)

The mathematics courses that we require all Techers to take are oriented towards giving future scientists and engineers and its ultimate goal is to give you a taste of what it means to *think* like a mathematician, so that you can take this out of your mental toolbox and use it when you need it in life. The task won't be easy, but it certainly can be done, and we can do so we have some fun along the way.

My name is Brian, and I'll be your Lead TA for Math 1a, Sec. 1 this quarter. Since this is a fairly large class, we also have the assistance of Gahye, who will also hold office hours. Here are the office hours for this class.

Date: October 1, 2016.

- Prof. Hadian-Jazi: Wednesdays, 4–5pm, 258 Sloan
- Gahye: Saturdays, 4–5pm, 385 Sloan
- Brian: Sundays, 9–10pm, 158 Sloan

(Note the curious symmetry in our office numbers.) I will also be available right after recitation, if you want to talk about anything.

Homework are due Monday afternoons. The course policy is not to accept any unexcused late homeworks, but if you really need it, don't be afraid to ask for help from the Dean or somebody else in that capacity. You don't need to be sick or dying to ask for one. We're really here to help you out and learn.

My philosophy for recitations is as follows. I will be ruthlessly practical and emphasize the ideas and methods you need in order to do your homework. I will leave the development of the theory to the lectures, and instead emphasize the quick and dirty techniques that will help you consolidate the knowledge from this week and apply them to the assigned problems.

You don't need to be taking copious notes, because I will post some notes on my website:

`http://hwang.caltech.edu/ma1/`

It's better to stay engaged with the flow of the class than simply try to write down everything; I've found the best way is to simply take minimal notes to keep my attention, then try and fill in the details on my own later, referring to the "official" notes if I get stuck or confused.

2. Basic Proof Techniques

The biggest difference between mathematics and other fields of study is the notion of a mathematical proof. It's hard to say precisely what a proof is (give it a shot[1]), but as a first approximation, it is a chain of logical statements that start with your assumptions and end with your desired conclusion. Hopefully, by seeing examples, you will internalize what this really means.

In this section, we study four basic methods of proof. We will use all of these in this class. In all cases, we want to prove that "$P => Q$", that is, a statement of the form "If P, then Q."

2.1. Direct Proofs.

Direct proofs are, as the name suggests, the most obvious way to show that "$P => Q$". Namely,

(1) Assume that P is true.
(2) Use P to show that Q must be true.

Here's an example of a direct proof.

Proposition 2.1. *If m and n are consecutive natural numbers, then $m + n$ is odd.*

[1] but an answer is probably as unhelpful or as circular as defining what "mathematics" means

Proof. Since m and n are consecutive natural numbers, we can write $n = m+1$.Therefore, we have

$$m + n = m + (m+1) = 2m + 1.$$

Since m is a natural number, $2m + 1 = m + n$ is odd. $\square$

2.2. **Proof by Contradiction (Reductio ad absurdum, if you want to be fancy).** Here we want to prove "$P \Rightarrow Q$" in a slightly funny way.

(1) Assume that P is true.
(2) Assume that $\neg Q$ ("not Q") is true.
(3) Use P and $\neg Q$ to demonstrate a contradiction.

Here's an example of this in action.

Theorem 2.2. *There are two irrational numbers a and b such that a^b is rational.*

"Proof with inner monologue". We will prove our statement by contradiction. To do this, we first assume that the negation of our theorem holds. In other words, we start off our proof by assuming the following hypothesis:

*There are **no** irrational numbers a and b such that a^b is rational.*

What do we do from here? Well, let's try throwing in numbers we know to be irrational into the above statement! Specifically, let's try setting both a and b equal to $\sqrt{2}$, which we know is irrational. Our hypothesis then tells us that

$$\sqrt{2}^{\sqrt{2}} \text{ is irrational.}$$

OK. What do we do now? Well, the only thing we really have is our assumption, our knowledge that $\sqrt{2}$ is irrational, and our new belief that $\sqrt{2}^{\sqrt{2}}$ is **also** irrational. The only thing we can really do is pick $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, and apply our hypothesis again. However, this will work! On one hand, our we have that a^b is irrational by our hypothesis. On the other hand, we have that a^b is equal to

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is clearly rational. This is a contradiction! Therefore, we know that our hypothesis must be false: there must be a pair of irrational numbers a, b such that a^b is rational. $\square$

Remark 1. An interesting quirk of the above proof is that it didn't actually give us a pair of irrational numbers a, b such that a^b is rational! It simply told us that either

- $\sqrt{2}^{\sqrt{2}}$ is rational, in which case $a = b = \sqrt{2}$ is an example, or
- $\sqrt{2}^{\sqrt{2}}$ irrational, in which case $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ is an example,

but it never actually tells us which pair satisfies our claim! This is a weird property of proofs by contradiction: they are often **nonconstructive** proofs, in that they will tell you that a statement is true or false without necessarily giving you an example that demonstrates the truth of that statement.

Remark 2. Can you write the statement above as an "$P \Rightarrow Q$" statement?

2.3. **Proof by Contrapositive.** When do you want to use a proof by contrapositive? Sometimes, proving "$P \Rightarrow Q$" directly is tricky: maybe P is a really subtle condition to start from, and we would prefer to start working from the other end of this implication. How can we do this?

Via the **contrapositive**! Specifically, if we have a statement of the form $P \implies Q$, the contrapositive of this statement is simply the statement

$$\neg Q \Rightarrow \neg P.$$

The nice thing about the contrapositive of any statement is that it's logically equivalent to the original statement! For example, if our statement was "all Techers are adorable," the contrapositive of our claim would be the statement "all nonadorable things are not Techers." These two statements clearly express the same meaning – one just starts out by talking about Techers, while the other starts out by talking about nonadorable things. So, if we want to prove a statement $P \implies Q$, we can always just prove the contrapositive $\neg Q \Rightarrow \neg P$ instead, because they're the same thing! This can allow us to switch from relatively difficult starting points (situations where P is hard to work with) to easier ones (situations where $\neg Q$ is easy to work with.)

In summary, you can apply the technique of proof by contrapositive as follows:

(1) Assume $\neg Q$.
(2) Use $\neg Q$ to show that $\neg P$ holds.

Remark 3. As you can see, proof by contrapositive is a close cousin of the proof by contradiction. Indeed, one way to demonstrate contradiction in Step (3) of the proof by contradiction is to show that both P and $\neg P$ hold, which cannot occur!

To illustrate this, consider the following example:

Theorem 2.3. *If $n \equiv 2 \mod 3$ (i.e. that $n - 2$ is a multiple of 3), then n is not a square: in other words, we cannot find any integer k such that $k^2 = n$.*

Proof. A direct approach to this problem looks hard. Basically, if we were to prove this problem directly, we would take any $n \equiv 2 \mod 3$ – i.e. any n of the form $3m + 2$, for some integer m – and try to show that this can never be a square. Basically, we'd be looking at the equation $k^2 = 3m + 2$ and trying to show that there are no solutions to this equation, which looks pretty nasty.

Since we are mathematicians, when presented with a tricky-looking problem, our instincts should be to try to make it trivial: in other words, to attempt different proof methods and ideas until one seems to "fit" our question. Let's look at the contrapositive of our statement:

$$\textit{If } n \textit{ is a square, then } n \not\equiv 2 \mod 3.$$

Equivalently, because every number is equivalent to either $0, 1$, or $2 \mod 3$, we're trying to prove the following:

$$\textit{If } n \textit{ is a square, then } n \equiv 0 \textit{ or } 1 \mod 3.$$

This looks much easier! – the initial condition is really easy to work with, and the later condition is rather easy to check.

Now that we have some confidence in our ability to prove our theorem, we proceed with the actual work: take any square n, and express it as k^2, for some natural number k. We can break k into three cases:

(1) $k \equiv 0 \mod 3$. In this case, we have that $k \equiv 3m$ for some m, which means that $k^2 = 9m^2 = 3(3m^2)$ is also a multiple of 3. Thus, $k^2 \equiv 0 \mod 3$.
(2) $k \equiv 1 \mod 3$. In this case, we have that $k \equiv 3m + 1$ for some m, which means that $k^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$. Thus, $k^2 \equiv 1 \mod 3$.
(3) $k \equiv 2 \mod 3$. In this case, we have that $k \equiv 3m + 2$ for some m, which means that $k^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$. Thus, $k^2 \equiv 1 \mod 3$.

Therefore, we've shown that k^2 isn't congruent to 2 mod 3, for any k. So we've proven our claim! $\square$

2.4. **Proofs by Induction.** Sometimes, in mathematics, we will want to prove the truth of some statement $P(n)$ that depends on some variable n. For example:

- $P(n) =$ "The sum of the first n natural numbers is $\frac{n(n+1)}{2}$."
- $P(n) =$ "If $q \geq 2$, we have $n \leq q^n$.
- $P(n) =$ "Every polynomial of degree n has at most n roots."

For any fixed n, we can usually use our previously-established methods to prove the truth or falsity of the statement. However, sometimes we will want to prove that one of these statements holds for *every* value $n \in \mathbb{N}$. How can we do this?

One method for proving such claims for every $n \in \mathbb{N}$ is to use mathematical induction.

(1) Prove our statement in the **base case**, that is, show that $P(1)$ is true.
(2) (**Induction/Inductive step**) *Assume* that $P(k)$ holds, and use this show that $P(k + 1)$ holds.

The intuitive reason why this works is as follows. Since we've established the base case, by applying the induction step over and over again, we see that

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$$

and so prove our statement for all $n \in \mathbb{N}$. The real reason why this works is that the principle of mathematical induction is rooted in the well-ordering principle, which was briefly discussed in class.

You'll get a lot of practice with this proof technique on the homework, but just so you're not confused, we'll go through a very simple application of proof by induction.

Proposition 2.4. *If $q \geq 2$, then $n \leq q^n$ for all $n \geq 0$.*

Proof. Base case. For the $n = 0$ case, we have

$$0 \leq q^0 = 1,$$

so the statement holds in this case. For $n = 1$, we also have

$$1 \leq q^1 = q$$

since $q \geq 2$.

Inductive step. Assume that $k \leq q^k$ (the induction hypothesis). We want to show that $k + 1 \leq q^{k+1}$.

By adding 1 to each side of the equality in the induction hypothesis, we have

$$k + 1 \leq q^k + 1.$$

To prove our statement, it is enough to prove the following lemma.

> *Lemma.* We have $q^k + 1 \leq q^{k+1}$.
>
> *Proof of Lemma.* Since $q \geq 2$, we have
>
> $$1 + \frac{1}{q^k} \leq 2 \leq q.$$
>
> Thus, by multiplying through by q^k, we obtain
>
> $$q^k + 1 \leq q^{k+1}$$
>
> as desired.

Thus, by the lemma,

$$k + 1 \leq q^k + 1 \leq q^{k+1}$$

as desired.

By the principle of mathematical induction, the inequality holds for all $n \geq 0$. $\square$

3. A Puzzle

Here's an interesting logic puzzle, related to some of proof techniques. Clearly the statement is false, but where does it go wrong?

Proposition 3.1. *There exists a unicorn.*

"Proof.". To prove there is a unicorn, it is enough to prove the (possibly) stronger statement that there exists an existing unicorn (i.e. a unicorn which exists). Namely, if there exists an existing unicorn, then there must exist a unicorn.

There are exactly two possibilities:

(1) an existing unicorn exists.
(2) an existing unicorn does not exist.

The second possibility is contradictory (how could an existing unicorn not exist? Just as a blue unicorn is necessarily blue, an existing unicorn must necessarily exist). Thus the first possibility must hold: an existing unicorn must exist. □

What is the problem with this proof?