MA 1: SESSION 2

1. Feedback on the first homework assignment

1.1. **Why can't you just reduce the statement to something "obvious"?** There's a kind of proof that I saw a lot of people do on the homework assignments. I want to try and illustrate why it's wrong by showing why it doesn't imply what you'd hope it does, by showing how it allows you to "prove" the following false statement.

Proposition 1.1. *We have* $0 = 2$.

"Proof".

$$\begin{aligned} 0 = 2 &\Leftrightarrow x - 1 - (x-1) = x + 1 - (x-1) \\ &\Leftrightarrow x - 1 = x + 1 \\ &\Leftrightarrow (x-1)^2 = (x+1)^2 \\ &\Leftrightarrow -2x + x^2 + 1 = 2x + x^2 + 1 \\ &\Leftrightarrow -2x = 2x \\ &\Leftrightarrow 4x^2 = 4x^2 \end{aligned}$$

Thus, $0 = 2$. □

Obviously this is wrong, and there are many ways in which this is wrong.

The first error. There's no words! I have no idea what you're trying to do, and so it just looks like nonsense. Please explain what you're trying to do!

The more serious error, you can't "reduce" the statement to "something obvious" and call it a proof, because it can allow you to prove things that are not true.

Here's an even more distilled version of the same idea.

Proposition 1.2. *We have* $1 = 2$.

Proof. Multiply both sides by 0, so $0 = 0$. Thus, $1 = 2$. □

It's a variation on the idea that you can achieve true conclusions from initial hypotheses, but this doesn't say anything about whether the initial hypotheses were true or false.

> **Moral**: You can't use the statement you're trying to prove at any point of your proof.

We were pretty lenient about this grading-wise this time, but please don't do this in the future.

Date: October 8, 2016.

2. Sup and Inf

The supremum of a set is its least upper bound, and the infimum is its greatest lower bound. Whereas the minimum and maximum of a set are the smallest and greatest elements, *that are contained in the set.*

We use sup and inf because they exist for nonempty sets of real numbers that are bounded above and below, respectively. Contrast this with min and max: They may not exist.

What are sup, inf, min, and max of the following sets?

Example 1.

$$S = \{\frac{1}{n} : n \in \mathbf{N}\}.$$

$\sup(S) = 1$, $\inf(S) = 0$, $\max(S) = 1$, $\min(S)$ does not exist.

Example 2. The interval $(0, 5) \subset \mathbf{R}$. The sup is 5, the inf is 0, neither the max nor min exist.

Example 3. Consider $A = \{r \in \mathbf{R} \mid 0 \leq r^2 < 2\}$.

Let $x = \sup(A)$. (What is it?) To prove rigorously that supremum is what we think it is involves a little more work, but we can prove a slightly weaker statement is fairly easy.

Proposition 2.1. *The inequality $x^2 > 2$ is impossible. (That is, $x^2 \leq 2$.)*

Proof. Say for contradiction that $x^2 > 2$.

Claim. The number $y = \sqrt{2}$ is such that $y \leq x$ and $y^2 = 2$.

Every element $a \in A$ is such that $0 \leq a^2 < 2$. Since a and y are both nonnegative and $y > 1$, we have

$$a^2 \leq y^2 \Rightarrow a < y.$$

Thus, y is an upper bound of A such that $y^2 \leq 2$. Since $\sup(A) \leq y$, we see that $\sup(A)^2 \leq 2$. □

3. The Least Upper Bound Property

A very important property of the real numbers is the following axiom.

> **The least upper bound axiom (a.k.a. the completeness axiom)**: Every nonempty set $S \subseteq \mathbf{R}$ that is bounded above has a supremum.

This seems really obvious, but it's really key to many special properties of the real numbers.

Proposition 3.1. *(Intermediate Value Theorem) If $f : [a, b] \to \boldsymbol{R}$ is a continuous*[1] *function, and suppose that $f(a) < 0$ and $f(b) > 0$, then $f(c) = 0$ for some point $c \in [a, b]$.*

[1] We haven't talked about this notion in this class yet, but I'm sure you remember this concept from your previous calculus classes.

Proof. Consider the set

$$S = \{s \in [a,b] \mid f(x) < 0 \text{ for all } x \leq s\}$$

consisting of the initial segment of $[a,b]$ that takes negative values under f. Then b is an upper bound for S, and its least upper bound c is such that $f(c) = 0$. □

Why do we emphasize this the fact that the least upper bound property is an axiom? It's because this is not satisfied by all sets, but is something that is special to $\mathbf{R}$.

The rational numbers are pretty similar to $\mathbf{R}$, but do not satisfy the least upper bound property. For example, the set

$$S = \{x \in \mathbf{Q} : x^2 < 2\}$$

does not have a least upper bound in $\mathbf{Q}$. (What is the least upper bound of this set in $\mathbf{R}$?)

Proposition 3.2. *The set S of $\mathbf{Q}$ does not have a least upper bound (in $\mathbf{Q}$).*

Proof. For every $x \in S$, we need to show that we can find a $y \in S$ such that $x < y$.

To every positive rational number x, consider

$$y := x - \frac{x^2 - 2}{x + 2} = \frac{2x + 2}{x + 2}.$$

All such y are positive. Subtracting $\sqrt{2}$ from both sides gives

$$y - \sqrt{2} = \frac{2x + 2 - \sqrt{2}x - \sqrt{2}2}{x + 2} = \frac{(x - \sqrt{2})(2 - \sqrt{2})}{x + 2}.$$

So pick $x \in S$. The associated y above is positive and in S, but since $x^2 - 2 > 0$, we know $y > x$. □

4. Integration

What is an integral? One of the things that first struck me what I was taking a class like this was that what I learned in calculus in high school wasn't so much the theory of integration, but rather integration techniques (implicitly assuming the integrals already exist).

I only knew that integration is the inverse operation to differentiation, which we learned first. But here we follow the historical precedent, and learn integration first. So how can we define an integral without referring to differentiation?

Definition 4.1. Let $f : [a,b] \to \mathbf{R}$ be a bounded function. The **integral**

$$I(f) = \int_a^b f(x)\,dx$$

is the unique number I such that

$$\int_a^b s(x)\, dx \leq I \leq \int_a^b t(x)$$

for every pair of step functions $s(x) \leq f(x) \leq t(x)$ for all $x \in [a, b]$.

When such an I exists, we say the function is **integrable** on $[a, b]$.

The lower integral of f is the supremum of the integral of all step functions $s(x) \leq f(x)$. Similarly, the upper integral of f is the infimum of the integral of all step functions $f(x) \leq t(x)$.

If a function is integrable, the upper and lower integrals are equal. If the upper and lower integrals differ, then the function is not integrable. Indeed, this is the easiest way to show a function is not integrable.

Remark 1. These integrals are referred to as *Riemann* integrals. There are other theories of integration, some also for real functions, some for different classes of functions.

For most of the integrals that you come across, one can just use formal rules and properties of integrals to solve.

Example 4. Calculate $\int_{-2}^{2} 2 - |x|\, dx$.

Solution. Easy. It is enough to use the properties of integrals of polynomials to prove this statement. The answer is 4, and we can double-check by just finding the area under the graph. □

However, there are some occasions where you need to use the actual definition of the integral in order to calculate things. (You probably don't need to do this much work on your homework assignment, but this justifies why we use the definition above for integration instead of just something that is the "opposite of differentiation.")

Example 5. Consider the function $f : [0, 1] \to \mathbf{R}$ defined by

$$f(1/2) = 1 \text{ and } f(x) = 0 \text{ for all } x \neq 1/2.$$

Then f is integrable.

Proof. For any partition P of $[0, 1]$, we have $L(P, f) = 0$ and thus the lower integral is always 0.

We want to show that the upper integral is 0. Let $\{x_1, \ldots, x_n\}$ be a partition of $[0, 1]$ and $\frac{1}{2} \in [x_i, x_{i+1}]$ for some i. Thus,

$$U(P, f) \leq 1 \cdot \Delta(x_i, x_{i+1}) := x_{i+1} - x_i$$

Since we can choose a partition P such that $\Delta(x_i, x_{i+1})$ is as small as possible, the upper integral is 0. More precisely, given any $\epsilon > 0$, we can find a partition P_ϵ such that $U(P, f) < \epsilon$. Thus, f is integrable with $\int_0^1 f(x)\, dx = 0$. □

Thus, with some work, we can integrate functions that we wouldn't be able to solve using the standard techniques from an earlier calculus class.

The following function is another famous example of a function whose integral you need a the formal definition of integration above to compute. It's a variation of the problem that you have on your homework. It's called Thomae's function (a.k.a. the "popcorn" function).

Example 6. Consider the function $t : [0, 1] \to \mathbf{R}$

$$t(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & x \notin \mathbf{Q} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \in \mathbf{Q} \text{ where } p \text{ and } q \text{ are relatively prime.} \end{cases}$$

Proposition 4.2. *The function t is continuous at every irrational number but discontinuous at every rational number.*

We'll ignore this fact this we haven't discussed the notion of continuity yet, but note that we have a LOT of discontinuities, so there's no way one can expect to use standard integration techniques to solve this integral (or even show it's integrable).

Proposition 4.3. *The function t is integrable and $\int_0^1 t = 0$.*

Proof. The irrational numbers are dense in $\mathbf{R}$, namely, for any partition $P - \{x_0, \ldots, x_n\}$, there is an irrational in every interval $[x_{i-1}, x_i]$. Thus, the lower integral is $L(t, P) = 0$.

To prove that t is integrable, it is enough to show that for every $\epsilon > 0$, there is a partition P_ϵ such that the upper integral $U(t, P) < \epsilon$.

Let $A_n = \{x : t(x) \geq \frac{1}{n}\}$. IF $x \in A_n$, then $x = i/j$, where $i/j \leq n$. Note that A_n is finite.

Let $\epsilon > 0$. Pick n such that $\frac{1}{n} < \frac{\epsilon}{2}$. Choose a partition P_ϵ such that each point of A_n is in an interval $[x_{i-1}, x_i]$, where

$$\Delta x_i = x_i - x_{i-1} < \frac{\epsilon}{2|A_n|}.$$

Set $B = \{i : A_n \cap [x_{i-1}, x_i] \neq \emptyset\}$. Note that $|B| \leq A_n|$. Let M_i denote max of f on the ith part of the partition P.

If $i \in B$, then $M_i < \frac{1}{n} < \frac{\epsilon}{2}$; on the other hand, if $i \notin B$, then $M_i = 1$. Thus,

$$\begin{aligned} U(t, P_\epsilon) &= \sum_{i \in B} M_i \Delta x_i + \sum_{i \notin B} M_i \Delta x_i \\ &< \sum_{i \in B} \frac{\epsilon}{2} \Delta x_i + \sum_{i \notin B} \Delta x_i \\ &< \frac{\epsilon}{2} + |A_n| \frac{\epsilon}{2|A_n|} \\ &= \epsilon. \end{aligned}$$

□

5. Non-integrable functions

The classic example of a function that is not Riemann integral is $f(x) = \frac{1}{x}$ on the interval $[0, 1]$. This fails to be integrable because f goes to infinity extremely quickly as $x \to 0$, so the area under the graph of this function is infinite. Another example is $f(x) = \frac{1}{x^2}$ in any interval containing 0.

However, it's not enough to say that functions with vertical asymptotes are not integrable, because the function $g(x) = \frac{1}{\sqrt{x}}$ has an asymptote at $x = 0$, just like $f(x)$ above, but we have

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = [2\sqrt{x}]\Big|_0^1 = 2.$$

Another classical example of a function that is not integrable is given on your homework.

Proposition 5.1. *If a function (over some domain) is Riemann-integrable (the integrals of this class), then its integral (over that domain) is finite.*

The contrapositive of this statement is an another way to show a function is not integrable.

Nonetheless, there is a large class of functions that we know are integrable.

Theorem 5.2. *(Apostol Theorem 1.12) Any non-decreasing bounded function $f : [a, b] \to \boldsymbol{R}$ is integrable.*

Proof. Let f be a monotonically increasing function. To show it's integrable, we need to show that for every $\epsilon > 0$, there exists a partition such that the difference between upper and lower integrals is less than ϵ. Choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. Then the max over the ith part is $f(x_i)$ and the min over an ith part is $f(x_{i-1})$. Thus,

$$U(P, f) - L(P, f) = \frac{b-a}{n} \sum_1^n [f(x_i) - f(x_{i-1})] = \frac{b-a}{n}[f(b) - f(a)] < \epsilon$$

for large n. □

It's not hard to construct a nondecreasing function with countably many discontinuities.

Example 7. Let $f : [0, 1] \to \mathbf{R}$ be the function

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 1 - \frac{1}{n}, & \text{if } 1 - \frac{1}{n} \le x < 1 - \frac{1}{n+1}. \end{cases}$$

This is integrable. Clearly it's discontinuous at $1 - \frac{1}{n}$ for all $n \in \mathbf{N}$.

(Challenge: What is the integral?)

It can also be the case that the integral exists, but there are no antiderivatives. Here's a famous example: the integral of the Gaussian. It is a theorem of Liouville that e^{-x^2} has no antiderivative, but it has an integral.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

You don't need to know how to do this (technically, it's multivariable calculus, but the tricks needed in this proof illustrate why the integration tricks in, say, AP/IB calculus don't suffice in general).

Proof. The function is even, so $I = 2\int_0^\infty e^{-x^2}\, dx$. We have

$$I^2 = 4\int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2}\, dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

The is a double integral over the first quadrant.

We then change to polar coordinates $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$:

$$\begin{aligned} I^2 &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, dr\theta \\ &= 4 \cdot \int_0^\infty re^{-r^2}\, dr \cdot \int_0^{\pi/2} d\theta \\ &= 2\pi \cdot \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^\infty \\ &= 2\pi \cdot \frac{1}{2} = \pi. \end{aligned}$$

By taking square roots, we get $I = \sqrt{\pi}$. □

So what's the point of all this? We want to decouple two things that we have conflated in our first encounter with calculus:

(1) Showing the function is integrable.
(2) Calculating the integral.

Of course, calculating the integral and showing it's finite implies integrability. But as we saw, for certain pathological functions, calculating the integral is very tricky, and we must use the formal definition of integral to rigorously show that the integral is a certain value.

The value of the definition of integration given above is that it applies to an extremely large class of functions, which do not even have to be continuous. While calculus is best-behaved when we restrict the study of continuous functions $f : \mathbf{R} \to \mathbf{R}$, it is remarkable that integration can be defined both without referring to differentiation and can apply to a far more general class of functions. Since most natural phenomena are non-linear and most objects in the world are rough and fractal-like, to attack the problems in the world beyond the ideal approximations and canned problems we see in our math and physics classes, we will need to use these more robust theories that extend our intuition from calculus. Ultimately, this is the real reason why we emphasize these difficult definitions of integration and $\epsilon - \delta$ definitions: it allows us to train ourselves to work at a higher level of abstraction and be better equipped to attack problems “as they are,” without having to settle for mere approximations.