MA 1: SESSION 3

1. Administrative announcement

For this week only, my office hours will be on Saturday at 6pm (1st floor), and Gahye's office hours will be Sunday at 8pm (3rd floor).

2. Homework 2 Feedback

A common mistake I saw for the first problem on the homework was an argument saying something along the lines of this false statement:

> *Myth:* If $\sum_{n=0}^{\infty} a_n$ is a sequence of real numbers such that $a_n \to \infty$ as $n \to \infty$, then the sequence converges.

This course is not really about convergence of such series (that would be the sequel to this course, Ma 1d), but it's important that you don't make make arguments like this. Let's go over the standard example of where this fails.

Proposition 2.1. *The series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

We have $\frac{1}{n} \to 0$ as $n \to \infty$, but this series does not converge.

Proof. There are many proofs, but here's a short one. Suppose for contradiction that the series converges to S, that is,

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

Then

$$\begin{aligned} H &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \frac{1}{2} + H, \end{aligned}$$

a contradiction. □

Also, note that the limit of problem 1(b) on the second homework is just the the exponential:

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

Moral: If you see a plausible-looking statement that you're not sure is true, try and prove the result yourself, or look it up in Apostol.

Date: October 15, 2016.

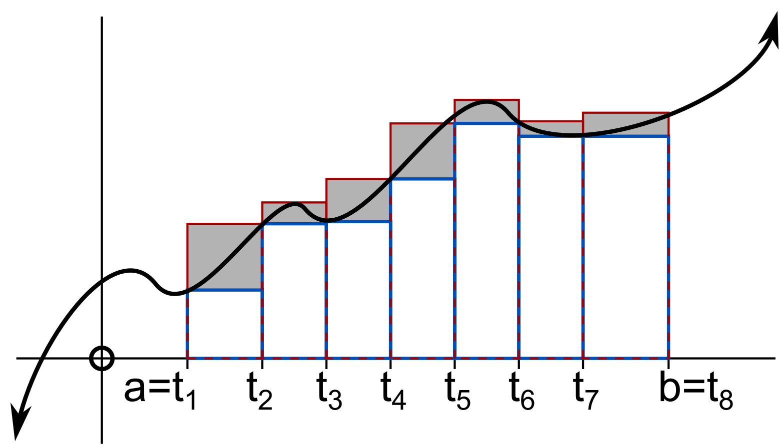
We mentioned integration last time, but there seemed to be some confusion with reconciling the familiar integrals from high school with the formal definition we'll use in this class. So here's another way of explaining it. This definition is equivalent to the other one.

Definition 3.1. A function f is **integrable** on the interval $[a, b]$ if and only if the following holds:

- For any $\epsilon > 0$,
- there is a partition $a = t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ of the interval $[a, b]$ such that

$$\left(\sum_{i=1}^{n} \sup_{x \in (t_i, t_{i+1})} (f(x)) \cdot (t_{i+1} - t_i) - \sum_{i=1}^{n} \inf_{x \in (t_i, t_{i+1})} (f(x)) \cdot (t_{i+1} - t_i) \right) < \epsilon$$

One way to interpret the sums above is through the following picture:



Specifically,

- think of the $(\sum \inf)$-sum as the area of the blue rectangles in the picture below, and
- think of the $(\sum \sup)$-sum as the area of the red rectangles in the picture below.
- Then, the difference of these two sums can be thought of as the area of the gray-shaded rectangles in the picture above.
- Thus, we're saying that a function $f(x)$ is **integrable** iff we can find collections of red rectangles – an "upper limit" on the area under the curve of $f(x)$ – and collections of blue rectangles – a "lower limit" on the area under the curve of $f(x)$ – such that the area of these upper and lower approximations are arbitrarily close to each other.

Note that the above condition is equivalent to the following claim: if $f(x)$ is integrable, we can find a sequence of partitions $\{P_n\}$ such that "the area of the gray rectangles with respect to the P_n partitions goes to 0" – i.e. a sequence of partitions $\{P_n\}$ such that

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sup_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i)-\sum_{i=1}^{n}\inf_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i)\right)=0.$$

In other words, there's a series of partitions P_n such that these upper and lower sums both converge to the same value: i.e. a collection of partitions P_n such that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sup_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i)=\lim_{n\to\infty}\sum_{i=1}^{n}\inf_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i).$$

If this happens, then we define

$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n}\sup_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i)=\lim_{n\to\infty}\sum_{i=1}^{n}\inf_{x\in(t_i,t_{i+1})}(f(x))\cdot(t_{i+1}-t_i),$$

and say that this quantity is the **integral** of $f(x)$ on the interval $[a,b]$. For convenience's sake, denote the upper sums of $f(x)$ over a partition P as $U(f(x),P)$, and the lower sums as $L(f(x),P)$.

This discussion, hopefully, motivates why we often say that the integral of some function $f(x)$ is just "the area under the curve" of $f(x)$. Pictorially, we are saying that a function is integrable if and only if we can come up with a well-defined notion of area for this function; in other words, if sufficiently fine upper bounds for the area beneath the curve (the ($\sum$ sup)-sums) are arbitrarily close to sufficiently fine lower bounds for the area beneath the curve (the ($\sum$ inf)-sums.)

The definition of the integral, sadly, is a tricky one to work with: the sups and infs and sums over partitions amount to a ton of notation, and it's easy to get lost in the symbols and have no idea what you're actually manipulating. If you ever find yourself feeling confused in this way, just remember the picture above! Basically, there are three things to internalize about this definition:

- the area of the **red** rectangles corresponds to the upper-bound ($\sum$ sup)-sums,
- the area of the **blue** rectangles corresponds to the lower-bound ($\sum$ inf)-sums, and
- if these two sums can be made to be arbitrarily close to each other – i.e. the area of the gray rectangles can be made arbitrarily small – then we have a "good" idea of what the area under the curve is, and can say that $\int_a^b f(x)$ is just the limit of the area of those red rectangles under increasingly smaller partitions (which is also the limit of the area of the blue rectangles.)

As we've seen, the integral is a difficult thing to work with using just the definition: later in this class, we'll develop more tools to help us actually do nontrivial things with the integral. To illustrate how working with the definition goes, though, let's work two examples:

3.1. **Calculating the Integral.**

Example 1. The integral of any constant function $f(x) = C$ from a to b exists; furthermore,

$$\int_a^b C\,dx = C \cdot (b-a).$$

Proof. Pick any constant function $f(x) = C$. To use our definition of the integral, we need to find a sequence of partitions P_n such that $\lim_{n\to\infty} U(f(x), P_n) - L(f(x), P_n)$ goes to 0. How can we do this? Well: what kinds of partitions of $[a, b]$ into n parts even exist?

One partition that often comes in handy is the uniform partition, where we break $[a, b]$ into n pieces all of the same length: i.e. the partition

$$P_n = \left\{ a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, a + 3\frac{b-a}{n}, \ldots a + n\frac{b-a}{n} = b. \right\}$$

In almost any situation where you need a partition, this will work excellently! In particular, one advantage of this partition is that the lengths $(t_{i+1} - t_i)$ in the upper and lower sums are all the same: they're specifically $\frac{b-a}{n}$.

Let's see what this partition does for our integral. If we look at $U(f(x), P_n)$, where P_n is the uniform partition defined above, we have

$$\begin{aligned}
U(f(x), P_n) =& \sum_{i=0}^{n-1} \left(\sup_{x \in \left(a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}\right)} f(x) \right) \cdot \left(a + (i+1)\frac{b-a}{n} - a - i\frac{b-a}{n} \right) \\
=& \sum_{i=0}^{n-1} \left(\sup_{x \in \left(a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}\right)} f(x) \right) \cdot \left(\frac{b-a}{n} \right) \\
=& \sum_{i=0}^{n-1} C \cdot \left(\frac{b-a}{n} \right), \text{ because } f(x) \text{ is a constant function,} \\
=& C\frac{b-a}{n} + C\frac{b-a}{n} + \ldots + C\frac{b-a}{n} \\
=& C \cdot (b-a).
\end{aligned}$$

Similarly, if we look at $L(f(x), P_n)$, we have

$$
\begin{aligned}
L(f(x), P_n) =& \sum_{i=0}^{n-1} \left(\inf_{x \in \left(a + i\frac{b-a}{n}, a+(i+1)\frac{b-a}{n}\right)} f(x) \right) \cdot \left(a + (i+1)\frac{b-a}{n} - a - i\frac{b-a}{n} \right) \\
=& \sum_{i=0}^{n-1} \left(\inf_{x \in \left(a + i\frac{b-a}{n}, a+(i+1)\frac{b-a}{n}\right)} f(x) \right) \cdot \left(\frac{b-a}{n} \right) \\
=& \sum_{i=0}^{n-1} C \cdot \left(\frac{b-a}{n} \right), \text{ because } f(x) \text{ is a constant function,} \\
=& C\frac{b-a}{n} + C\frac{b-a}{n} + \ldots + C\frac{b-a}{n} \\
&= C \cdot (b-a).
\end{aligned}
$$

Therefore, the limit

$$
\lim_{n\to\infty} U(f(x), P_n) - L(f(x), P_n) = \lim_{n\to\infty} C \cdot (b-a) - C \cdot (b-a) = \lim_{n\to\infty} 0 = 0,
$$

and consequently our integral exists and is equal to

$$
\lim_{n\to\infty} U(f(x), P_n) = C \cdot (b-a).
$$

□

OK, that one was pretty simple, but we got to see how we can use all the moving parts of the definition of the integral. Let's try a slightly trickier example.

Example 2. The function $f(x) = x^p$ is integrable on $[0, b]$ for any $p \in \mathbb{N}$ and $b \in \mathbb{R}^+$. Furthermore, the integral of this function is $\frac{b^{p+1}}{p+1}$.

Proof. Our uniform partition, where we broke our interval up into n equal parts, worked pretty well for us above! Let's see if it can help us in this problem as well. If we let $P_n = \{0, \frac{b}{n}, 2\frac{b}{n}, \ldots n\frac{b}{n}\} = b$, then $U(f(x), P_n)$ is just

$$
\sum_{k=0}^{n-1} \sup_{x \in \left(k\frac{b}{n}, (k+1)\frac{b}{n}\right)} (x^p) \cdot \left((k+1)\frac{b}{n} - k\frac{b}{n} \right) = \sum_{k=0}^{n-1} \left((k+1)\frac{b}{n} \right)^p \cdot \frac{b}{n} = \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} (k+1)^p,
$$

and that the lower-bound sum, ($\sum \inf$), is

$$
\sum_{k=0}^{n-1} \inf_{x \in \left(k\frac{b}{n}, (k+1)\frac{b}{n}\right)} (x^p) \cdot \left((k+1)\frac{b}{n} - k\frac{b}{n} \right) = \sum_{k=0}^{n-1} \left(k\frac{b}{n} \right)^p \cdot \frac{b}{n} = \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n} k^p.
$$

Taking the limit of their difference, we obtain

$$\begin{aligned}
\lim_{n\to\infty} U(f(x),P_n) - L(f(x),P_n) &= \lim_{n\to\infty} \left(\left(\frac{b^{p+1}}{n^{p+1}} \sum_{k=0}^{n-1} (k+1)^p \right) - \left(\frac{b^{p+1}}{n^{p+1}} \sum_{k=0}^{n-1} k^p \right) \right) \\
&= \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}} \left(\left(\sum_{k=0}^{n-1} (k+1)^p \right) - \left(\sum_{k=0}^{n-1} k^p \right) \right) \\
&= \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}} \left((1^p + 2^p + 3^p + \ldots + n^p) - (0^p + 1^p + 2^p + \ldots + (n-1)^p \right. \\
&= \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}} \left(n^p \right) \\
&= \lim_{n\to\infty} \frac{b^{p+1}}{n} \\
&= 0.
\end{aligned}$$

Thus, by our definition, the function x^p is integrable on $[0,b]$! Furthermore, the integral of this function is just

$$\lim_{n\to\infty} U(f(x),P_n) = \lim_{n\to\infty} \left(\frac{b^{p+1}}{n^{p+1}} \sum_{k=0}^{n-1} (k+1)^p \right) = \lim_{n\to\infty} \frac{b^{p+1}}{n^{p+1}} \left(\sum_{k=1}^{n} (k)^p \right).$$

So: it suffices to understand what the sum $(\sum_{k=1}^{n} (k)^p)$ is, for any p. Unfortunately, doing **that** is rather hard. We can find formulae for a few small cases: when $p = 1$, for example, using an identity we've proven before, this is just

$$\lim_{n\to\infty} \frac{b^2}{n^2} \left(\sum_{k=1}^{n} k \right) = \lim_{n\to\infty} \frac{b^2}{n^2} \cdot \left(\frac{n(n+1)}{2} \right) = \lim_{n\to\infty} \frac{b^2}{2} \cdot \frac{n(n+1)}{n^2} = \frac{b^2}{2},$$

which indeed is the integral of x from 0 to b. □

In general, though, finding these sums is hard! Later, we'll develop some tricks to get around this problem.

4. Applications of Integrals

Most of the problems on the homework this week are fun problems, each involving a different application of integration. The techniques to solve the integral should be easy and familiar, but the hard part is setting up the integral correctly. While we don't have time to go over all the cases, let's focus on a couple of applications that give the feeling of how one might go about solving these kinds of problems.

4.1. **Polar coordinates.** Recall how polar coordinates are defined. In place of the usual $(x, y) \in \mathbf{R}^2$, we have

$$r^2 = x^2 + y^2, \quad 0 \leq \theta \leq 2\pi,$$

where we convert back to Euclidean coordinates via

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The area formula for the polar coordinate function $r(\theta)$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

Let's apply this to a specific example.

Example 3. Find the area bounded by $r(\theta) = 2 - 2\sin\theta$.

Solution. We first note a couple of things. We know that the maximum value of $\sin\theta$ is 1 and the minimum value is -1, and so we know that the graph of our function is contained in a circle of radius > 4 from the origin. We can then graph it and see that it is symmetric with respect to the y-axis, so we can take $\theta \in [-\pi/2, \pi/2]$ and then multiple the area of that integral by 2.

Thus, the area bounded by $r(\theta)$ is given by

$$\begin{aligned}
A &= 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}(2 - 2\sin\theta)^2 \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} (4 - 8\sin\theta + 4\sin^2\theta) \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4 - 8\sin\theta + 2(1 - \cos 2\theta) \, d\theta \quad [\text{since } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)] \\
&= \int_{-\pi/2}^{\pi/2} 6 - 8\sin\theta - 2\cos 2\theta \, d\theta \\
&= [6\theta - 8\cos\theta - \sin 2\theta]\Big|_{-\pi/2}^{\pi/2} \\
&= 6\pi.
\end{aligned}$$

□

4.2. **Volume of a solid of revolution.** If we revolve a function around the x-axis, we get a solid of revolution that we can evaluate by integrating along x.

How can we do this? Well, each cross-section of such a 3d solid cut by a plane perpendicular to the x-axis is a circular disk with radius $f(x) = y$. Thus, the area cut out by this slice at x is

$$\pi r^2 = \pi(f(x))^2.$$

The volume is then given by

$$V = \int_a^b \pi(f(x))^2\,dx.$$

Example 4. Find the volume of an ellipsoid with major axis $2a$ and minor axis $2b$ (where both $a, b > 0$, of course). Since integration is translation-invariant, we can assume that the ellipsoid is centered at the origin. We need an expression for y^2.

The equation for the ellipse with the given axes is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus, multiply through by a^2b^2, we get

$$b^2x^2 + a^2y^2 = a^2b^2.$$

To isolate the y^2 on side, we get

$$a^2y^2 = a^2b^2 - b^2x^2 = b^2(a^2 - x^2).$$

Thus,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

We note that the ellipsoid is symmetric about x, so it is enough to find the volume of half of it and double. In other words, the volume is

$$\begin{aligned} V &= 2 \cdot \pi \int_0^a y^2\,dx \\ &= 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx \\ &= 2\pi \frac{b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]\Big|_0^a \\ &= 2\pi \frac{b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right] \\ &= 2\pi \frac{b^2}{a^2}\frac{2a^3}{3} \\ &= \frac{4}{3}\pi b^2 a. \end{aligned}$$

Whenever we've derived a formula like this, it's good to check the value makes sense for some predicted values, or gives us back something that we know. For instance, we check that if we set $a = b =: r$, we get an ellipsoid with the minor and major axis of length $2r$, in other words, a sphere. Indeed, by plugging in the result into the formula we obtained, we get

$$V = \frac{4}{3}\pi r^3,$$

the familiar formula for the volume of the sphere.