MA 1: SESSION 4

1. The Definition of a Limit

The first thing we need to do is to give a rigorous definition of what the limit is.

Definition 1.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of (real) numbers. We say that

$$\lim_{n\to\infty} a_n = L$$

if for all $\epsilon > 0$, there exists an $N = N_\epsilon \in \mathbf{N}$ such that for all $n > N$, we have $|a_n - L| < \epsilon$.

The simplest way to show that a sequence converges is sometimes just to use the definition of convergence, that is, you want to show that for any distance $\epsilon > 0$, you can force the a_n's to be within ϵ of our limit, for n sufficiently large.

How should we use the definition? Here's one method.

- Examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on n and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
- Using this simple upper bound, given $\epsilon > 0$, determine a value of N such that whenever $n > N$, our simple bound is less than ϵ. This is usually pretty easy: because these simple bounds go to 0 as n gets large, there's always some value of N such that for any $n > N$, these simple bounds are as small as we want.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $n > N$, $|a_n - L| < \epsilon$.

Example 1. Show that

$$\lim_{n\to\infty} \left[\sqrt{n+1} - \sqrt{n}\right] = 0.$$

Solution. Let's try and prove convergence directly using our method above: (a) start with $|a_n - L|$, (b) try to find a simple upper bound on this quantity depending on n, and (c) use this simple bound to find for any ϵ a value of N such that whenever $n > N$, we have

$$|a_n - L| < (\text{simple upper bound}) < \epsilon.$$

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Let's look at the quantity $|\sqrt{n+1}-\sqrt{n}-0|$:

$$\begin{aligned}|\sqrt{n+1}-\sqrt{n}-0| &= \sqrt{n+1}-\sqrt{n} \\ &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &< \frac{1}{\sqrt{n}}.\end{aligned}$$

All we did here was hit our $|a_n - L|$ with some algebra, and kept trying things until we got something simple. The specifics aren't as important as the idea here: just start with the $|a_n - L|$ bit, and try everything until it's bounded by something simple and small!

In our specific case, we've acquired the upper bound $\frac{1}{\sqrt{n}}$, which looks rather simple: so let's see if we can use it to find a value of N.

Take any $\epsilon > 0$. If we want to make our simple bound $\frac{1}{\sqrt{n}} < \epsilon$, this is equivalent to making $\frac{1}{\epsilon} < \sqrt{n}$, i.e $\frac{1}{\epsilon^2} < n$. So, if we pick $N > \frac{1}{\epsilon^2}$, we know that whenever $n > N$, we have $n > \frac{1}{\epsilon^2}$, and therefore that our simple bound is $< \epsilon$. But this is exactly what we wanted!

In specific, for any $\epsilon > 0$, we've found a N such that for any $n > N$, we have

$$|\sqrt{n+1}-\sqrt{n}-0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = 0$. □

Let's see how you should write this up.

Proof. Let $\epsilon > 0$. Pick $N > \frac{1}{\epsilon^2}$. Then for all $n > N$, we have

$$|\sqrt{n+1}-\sqrt{n}-0| = \text{[algebra steps from above]} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon.$$

□

That's all you need.

For sequences, we're taking the limit over natural numbers, but what we're really interested in are limits for functions.

Definition 1.2. Given a function $f : \mathbf{R} \to \mathbf{R}$, we say that

$$\lim_{x\to c} f(x) = L$$

if for any $\epsilon > 0$ there exists a $\delta = \delta_\epsilon > 0$ such that for all x such that $|x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

The definition is just the natural generalization of the notion for sequences above.

Here are some general facts about limits (they also apply to limits of sequences).

Proposition 1.3. *("Arithmetic of limits") [Apostol Theorem 3.1] Let f and g be functions such that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then*

(1) $\lim_{x\to p}[f(x) + g(x)] = A + B$,
(2) $\lim_{x\to p}[f(X) - g(x)] = A - B$,
(3) $\lim_{x\to p} f(x) \cdot g(X) = A \cdot B$,
(4) $\lim_{x\to p} f(x)/g(x) = A/B$ *if* $B \neq 0$.

2. Continuous Functions

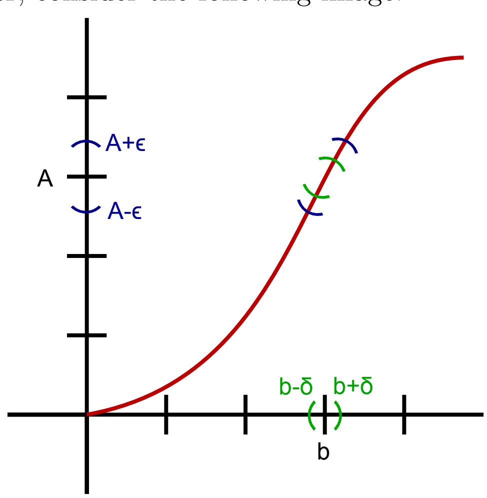
With the definition of limits of functions at certain points in hand, we can now define the fundamental objects of study in a single-variable calculus course.

Definition 2.1. A function $f : \mathbf{R} \to \mathbf{R}$ is **continuous at** $a \in \mathbf{R}$ if for every $\epsilon > 0$ there exists a $\delta = \delta_\epsilon > 0$ such that for all $x \in \mathbf{R}$ such that $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. In other words, if $\lim_{x\to a} f(x) = f(a)$.

We say that f is **continuous on** $[a, b]$ if it is continuous at all $x \in [a, b]$. If $[a, b] = \mathbf{R}$, we say simply that f is **continuous**.

If you haven't seen this definition before, it's a lot of absorb at once. But I'll try and illustrate it.

This definition is a little strange, isn't it? At least, the $\epsilon - \delta$ "rigorous" definition is somewhat strange: how do these weird symbols connect with the rather simple concept of "as x approaches a, $f(x)$ approaches $f(a)$"? To see this a bit better, consider the following image:



An easy corollary of the "arithmetic of limits" properties we mentioned before is the following result.

Proposition 2.2. *If f and g are functions that are continuous at p, then the sum $f+g$, the difference, $f-g$, and the product $f \cdot g$ are continuous at p. Furthermore, if $g(p) \neq 0$, the quotient f/g is also continuous at p.*

The graph above shows pictorially what's going on in our "rigorous" definition of limits and continuity: essentially, to rigorously say that "as x approaches a, $f(x)$ approaches $f(a)$", we are saying that

- for any distance ϵ around $f(a)$ that we'd like to keep our function,
- there is a neighborhood $(a-\delta, a+\delta)$ around a such that
- if f takes only values within this neighborhood $(a-\delta, a+\delta)$, it stays within ϵ of $f(a)$.

Basically, what this definition says is that if you pick values of x sufficiently close to a, the resulting $f(x)$'s will be as close as you want to be to $f(a)$ – i.e. that "as x approaches a, $f(x)$ approaches $f(a)$."

This, hopefully, illustrates what our definition is trying to capture – a concrete notion of something like convergence for *functions*, instead of sequences of real numbers. So: how can we prove that a function f has some given limit L? Motivated by this analogy to sequences, we have the following method to prove that $\lim_{x\to a} f(x) = L$ straight from the definition:

(1) First, examine the quantity

$$|f(x) - L|.$$

Using algebra/cleverness, try to find a simple upper bound for this quantity of the form

$$(\text{things bounded when } x \text{ is near } a) \cdot (\text{function based on} |x-a|).$$

Some sample candidates: things like $|x-a| \cdot (\text{constants})$, or $|x-a|^3 \cdot (\text{bounded functions like } \sin(x))$.

(2) Take your bounded part, and **bound** it! In other words, find a constant bound $C > 0$ and a value $\delta_1 > 0$ such that whenever x is within δ_1 of a, we have

$$(\text{bounded things}) < C.$$

(3) Take your function based on $|x-a|$ and your constant C from the above step, and starting from the equation

$$(\text{function based on } |x-a|) < \frac{\epsilon}{C},$$

solve for $|x-a|$ in terms of ϵ and C, by performing only reversible steps. This then gives you some equation of the form

$$|x-a| < (\text{thing in terms of } C, \epsilon\text{'s}).$$

Define δ_2 to be this "thing in terms of C, ϵ's."

(4) Let $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have just proven that we satisfy both the equations

$$\text{(bounded things)} < C, \qquad \text{and}$$
$$\text{(function in } |x-a|) < \frac{\epsilon}{C}.$$

If we combine these observations with the simple bound we derived in our first step, we've proven that whenever $|x-a| < \delta$, we have

$$|f(x) - L| < \text{(bounded things)}(|x-a| \text{ things}) < C \cdot \frac{\epsilon}{C} = \epsilon.$$

But this is exactly what we wanted to prove – this is the $\epsilon - \delta$ definiton of a limit! So we are done.

The following example ought to illustrate what we're talking about here:

Example 2. The function $\frac{1}{x^2}$ is continuous at every point $a \neq 0$.

Proof. We want to prove that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$, for any $a \neq 0$.

We proceed according to our blueprint:

(1) First, we examine the quantity $\left|\frac{1}{x^2} - \frac{1}{a^2}\right|$:

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \left|\frac{a^2}{a^2x^2} - \frac{x^2}{a^2x^2}\right| \\
&= \left|\frac{a^2 - x^2}{a^2x^2}\right| \\
&= \left|\frac{(a-x)(a+x)}{a^2x^2}\right| \\
&= |a-x| \cdot \left|\frac{(a+x)}{a^2x^2}\right| \\
&= |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right|.
\end{aligned}$$

By algebraic simplification, we've broken our expression into two parts: one of which is $|x-a|$, and the other of which is bounded near $x = a$.

For values of x rather close to a, because $a \neq 0$, we can bound this as follows: pick x such that x is within $a/2$ of a. Then we have

$$\begin{aligned}
\left|\frac{(a+x)}{a^2x^2}\right| &\leq \left|\frac{(a+(3a/2))}{a^2x^2}\right| \\
&\leq \left|\frac{(a+(3a/2))}{a^2(a/2)^2}\right| \\
&= \left|\frac{10}{a^3}\right|
\end{aligned}$$

which is some nicely bounded constant. So, when we pick our δ, if we just make sure that $\delta < a/2$, we know that we have this quite simple and excellent upper bound

$$\left|\frac{(a+x)}{a^2x^2}\right| < \left|\frac{10}{a^3}\right|.$$

(2) So: we have bounded the bounded part by $\left|\frac{10}{a^3}\right|$. Now, we want to take the remaining $|x-a|$ part, which is exactly $|x-a|$, and solve the equation

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}$$

for $|x-a|$, given any arbitrary $\epsilon > 0$. Conveniently, this is already done! In fact, if we're using our blueprint and we can make our "function in terms of $|x-a|$" precisely $|x-a|$, this is always this easy. Therefore, if we set $\delta_2 = \frac{a^3\epsilon}{10}$, then whenever $|x-a| < \delta_2$, we have

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}.$$

(3) Now, set $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have

$$|f(x) - L| \leq |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right| < \frac{10}{a^3} \cdot \frac{a^3\epsilon}{10} = \epsilon,$$

which is precisely what we needed to show to satisfy the $\epsilon - \delta$ definition of a limit. Therefore, we have proven that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$ for any $a \neq 0$, as desired.

$\square$