MA 1: SESSION 5

1. Announcements

You should have picked up your midterms by now. If not, go to the math office immediately tomorrow, and make sure that they have copies, or ask Prof. Hadian.

The midterm is open book, open notes, and you may refer to the problem statements for homeworks, but not your own solutions or the solutions posted online. You also may not discuss the exam with anybody once you begin (except Prof. Hadian), including me, so take that into consideration before you begin.

2. Epsilon-Delta Proofs

Example 1. Suppose that $x, y \in \mathbf{R}$ have the property that $|x - y| < \epsilon$ for every $\epsilon > 0$. Show that $x = y$.

Proof. Suppose $x \neq y$. Then $|x - y| = a > 0$. We have a positive real number b such that $a > b > 0$. Since $b > 0$, by assumption $|x - y| < b$. But then we have a contradiction since this implies that

$$a = |x - y| < b.$$

Hence, we must have $x = y$. $\square$

Example 2. Use an ϵ-δ proof to show that $f(x) = x^3$ is continuous at $x = 2$.

Proof. In other words, we need to show that $\lim_{x\to 2} x^3 = 8$.

Let $\epsilon > 0$. We want to find a $\delta > 0$ such that for any $x \in \mathbf{R}$ with $0 < |x - 2| < \delta$ forces $|x^3 - 8| < \epsilon$. We claim that

$$\delta = \min\left\{\frac{\epsilon}{19}, 1\right\}$$

works. To see this, suppose that $|x - 2| < \delta$. Then

$$\begin{aligned}
|x^3 - 8| &= |x - 2||x^2 + 2x + 4| \\
&< |x - 2|19 \quad \text{since } |x^2 + 2x + 4| < 19 \text{ if } |x - 2| < 1 \\
&< \frac{\epsilon}{19} \cdot 19 \\
&= \epsilon.
\end{aligned}$$

$\square$

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This is what you should write. But how did we find such a value of δ? Of course, we worked backwards from $|x^3 - 8|$ and didn't choose a δ until we obtained an expression for $|x - 2|$. (See Session 4 notes.)

3. Integration

Proposition 3.1. *If $f : [a, b] \to \boldsymbol{R}$ is continuous, then it is integrable.*

Proof. To show f is integrable, we need to show that for all $\epsilon > 0$, we have

$$\inf_P U(P, f) - \sup_P L(P, f) < \epsilon$$

where the limit runs over all partitions P of $[a, b]$.

Apostol Theorem III.13 says that if f is continuous on $[a, b]$, then for every $\epsilon_0 > 0$, there is a partition of $[a, b]$ into a finite number of subintervals such that $\max(f) - \min(f)$ on each subinterval is less than ϵ_0.

Given a partition $P = \{x_1, \ldots, x_{n-1}\}$ We have

$$U(P, f) - L(P, f) = \sum_{i=0}^{n-1} \left(\sup_{x \in (x_i, x_{i+1})} f(x) - \inf_{x \in (x_i, x_{i+1})} f(x) \right) (x_{i+1} - x_i).$$

By the theorem above, give any $\epsilon_0 < 0$, there exists a partition $P_0 = P_{\epsilon_0}$ such that

$$U(P_0, f) - L(P_0, f) \leq \epsilon_0 \sum_{i=0}^{m-1} (x_{i+1} - x_i) = \epsilon_0 (b - a).$$

Thus, set $\epsilon_0 = \epsilon/(b - a) > 0$, which gives us a partition P'' such that $U(P'', f) - L(P'', f) < \epsilon$, showing that f is integrable. $\square$

4. Limit Manipulation

Example 3. Let x, y be a pair of positive real numbers such that $x < y$. Show that

$$\lim_{n \to \infty} (x^n + y^n)^{1/n} = y.$$

Proof. This time, let's apply the squeeze theorem. Specifically, notice that

$$y = (y^n)^{1/n} < (x^n + y^n)^{1/n} < (y^n + y^n)^{1/n} = 2^{1/n} y.$$

Therefore, because

$$\lim_{n \to \infty} y = y$$

and

$$\lim_{n \to \infty} 2^{1/n} y = y \cdot \lim_{n \to \infty} 2^{1/n} = y \cdot 1 = y,$$

the squeeze theorem tells us that

$$\lim_{n \to \infty} (x^n + y^n)^{1/n} = y$$

as well. $\square$

5. Thomae's Function

Example 4 (Thomae's Function a.k.a the popcorn function)**.** Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are relatively prime integers with } q > 0 \\ 0 & x \text{ is irrational} \end{cases}$$

Show that $f(x)$ is continuous at every irrational number and discontinuous at every rational number.

Proof. The question is asking us to prove two things: 1. when x is irrational, f is continuous at x; and 2. when x is rational, f is *not* continuous at x. Let's first prove that f is continuous at irrational numbers.

Let x be irrational. Then, we first note that $f(x) = 0$.

Now, let $\varepsilon > 0$. Choose a natural number m large enough so that $\frac{1}{m} < \varepsilon$. Then, x lies in a unique interval of the form $\left(\frac{k}{m}, \frac{k+1}{m}\right)$, say $x \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$.

Note that for all $n \leq m$, there can be at most 1 number of the form $\frac{r}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$. Why? Suppose $\frac{r}{n}, \frac{s}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$. Then, we have

$$\begin{array}{rrcl} & |\frac{r}{n} - \frac{s}{n}| & < & |\frac{K+1}{m} - \frac{K}{m}| \\ \Rightarrow & \frac{|r-s|}{n} & < & \frac{1}{m} \\ \Rightarrow & \frac{1}{n} & < & \frac{1}{m} \\ \Rightarrow & n & > & m \end{array}$$

But this contradicts our assumption that $n \leq m$. Hence, there is at most one number of the form $\frac{r}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$.

In particular, this implies that there are finitely many rational numbers $\frac{p}{q} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$ in the reduced form where $q \leq m$ (We showed above that, in fact, there can be at most m such numbers). Denote these numbers $p_1, \ldots, p_s$ and let

$$\delta = \min(|x - p_1|, \ldots, |x - p_s|)$$

We claim that for all y with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. We can divide this in two cases: Y can be irrational or rational.

Suppose y is an irrational number with $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |0 - 0| = 0 < \varepsilon$$

Now suppose y is a rational number with $|x - y| < \delta$. Since y is rational, we can write $y = \frac{p}{q}$. But since y is closer to x than any of $p_1, \ldots, p_s$, we conclude $q > m$ (we chose $\delta = \min(|x - p_1|, \ldots, |x - p_s|)$ precisely to make this happen!). Therefore, we have the following

$$|f(x) - f(y)| = \left|0 - \frac{1}{q}\right| = \frac{1}{q} < \frac{1}{m} < \varepsilon$$

Combining these two cases, we see that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Hence, f is continuous at x.

Now, let's prove that f is not continuous at every rational numbers.

Let $x = \frac{p}{q}$ be a rational number. Then, $f(x) = \frac{1}{q}$. To show that f is *not* continuous at x, we have to find *some* value of $\varepsilon > 0$ for which we cannot find a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Consider $\varepsilon = \frac{1}{q}$. I claim that for this value of ε, we can't find a value of δ satisfying the conditions of the definition.

Let's prove this by contradiction. Suppose there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. However, note that since $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains an irrational number, say $y \in (x - \delta, x + \delta)$. Now, for this y we have $|x - y| < \delta$. However, since y is irrational,

$$|f(x) - f(y)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q} \not< \varepsilon = \frac{1}{q}$$

This is a contradiction! Hence, we conclude that for $\varepsilon = \frac{1}{q}$, there does not exist $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Therefore, f is not continuous at every rational number. □

6. Theory Questions

Example 5. If f is differentiable at a, then it is continuous at a.

Proof. Recall that f is differentiable at a if f is defined in a neighborhood of a and

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. If this limit exists, we call it $f'(a)$.

If f is differentiable at a, then

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} f(a) + (x - a)\frac{f(x) - f(a)}{x - a} \\
&= f(a) + \left(\lim_{x \to a} x - a\right)\left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \\
&= f(a) + 0 \cdot f'(a) \\
&= f(a).
\end{aligned}$$

Hence, by the definition of continuity, f is continuous at a. □