MA 1: SESSION 6

1. Introduction: Differentiability and Integrability

We saw that things can get pretty bad with respect to integration. But can they be bad while staying relatively nice?

We know that differentiability implies continuity, and we'll later learn about the relation between differentiability and integrability (which you've used for nice functions in your previous calculus classes). But is there a function that is continuous without being differentiable?

Example 1. Of course. We can consider the function $f : \mathbf{R} \to \mathbf{R}$ with

$$f(x) = |x|.$$

It is continuous everywhere, even at 0, but it is not differentiable at 0. (Check!)

OK, but that was a function that was differentiable every point except at $x = 0$. Is there a function that is continuous everywhere but differentiable nowhere? Surprisingly, the answer turns out to be yes!

Example 2. This famous (class of) examples is due to Weierstrass. Before this example, it was widely believed that every continuous function was differentiable except on a set of isolated points.

For instance, a simple example of such a function is

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{1}{2}\sin(2x) + \frac{1}{4}\sin(4x) + \cdots .$$

This series converges for all x, because it's "absolutely convergent," indeed, the absolute value of this function is bounded by 2 at any point. It's continuous since it's a bounded sum of a continuous functions.

It turns out that this is continuous for all x, but not differentiable at any x.

This function is 2π-periodic, that is, $f(x) = f(x + 2\pi)$, and it looks very "bumpy" and so, you might guess it's not differentiable everywhere. However, you can prove this rigorously with some work.

There are many interesting things about such functions, but here are two in particular. First, as I indicated in the beginning, it turns out that "most" functions from $\mathbf{R}$ to $\mathbf{R}$ are some kind of Weierstrass function; and for functions from $\mathbf{R}^n \to \mathbf{R}^n$, tend to look like some kind of "Brownian motion" as you travel along a path. This is an example of the same phenomena that

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you experience with real numbers. “Most” real numbers are not rational, but almost all of the numbers you've seen in your life are rational.

Secondly, the Weierstrass function is an example of one of the first “fractals” studied, although this term wasn't in common usage until Mandelbrot popularized it in the late 20th century. Weierstrass functions have “detail at every level,” in that if you keep on zooming in onto the further, it does not ever get closer to being a straight line. More precisely, between any two points, the function is never monotone. This roughness, as Mandelbrot and others noticed, is a phenomenon that seems to be ubiquitous in nature, and it seems to be this kind of self-similarity that makes objects look “natural” (as opposed to artificial).

1.1. **Differentiability and Integrability Do Not Imply Each Other.**

This is an important point, but is something obscured in the slew of definitions. There are differentiable functions that are not integrable, and there are integrable but non-differentiable functions.

For instance,

$$f(x) = \frac{1}{x}$$

is differentiable in (0,1), but it's not integrable in this interval. Why? We have

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{x}\,dx = \ln 1 - \ln 0$$

which is undefined.

Another more prosaic example is simply the function $f(x) = x$. This is certainly differentiable, but it is not an integrable function from $\mathbf{R}$ to $\mathbf{R}$ since its integral is infinite. However, it IS integral, on, say, an interval.

A Weierstrass function is an example of an integrable but non-differentiable function.

2. Fundamental Theorem of Calculus

Here's the main result of the course. Essentially everything else in single-variable calculus is just a corollary of this result. This is the precise sense in which “integration and differentiation are inverse operations.”

Theorem 2.1. *(The Fundamental Theorem of Calculus, Version 1) Let F be a continuous function on the interval $I = [a, b]$. Suppose F is differentiable everywhere in the interior of the interval I (that is, on (a, b)) with derivative f, which is (Riemann) integrable. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

As you may remember from your previous calculus course, we usually say that F is the **antiderivative** or the **primitive** of f.

All the hypotheses for the theorem are necessary, if you relax any of them, one can easily produce a counterexample.

Remark 1. Note that we can only talk about differentiability on the *interior* of I, because in $\mathbf{R}$, the definition of the derivative at a point depends on limits coming from both the positive and negative sides.

We also commonly apply the following version of the Fundamental Theorem of Calculus. The following is a slightly souped-up version of the FTC presented in class, and comes from your textbook.

Theorem 2.2. *(The Fundamental Theorem of Calculus, Version 2) Let f be a function that is integrable on $[a, x]$ for each x in $[a, b]$. Let c be such that $a \leq c \leq b$ and define a new function*

$$F(x) = \int_c^x f(t)\, dt, \quad \text{if } a \leq x \leq b.$$

Then the derivative $F'(x)$ exists at each point x in the open interval (a, b) where f is continuous, and for such x, we have

$$F'(x) = f(x).$$

As a corollary of this second theorem, we obtain the following useful result that is also sometimes referred to as a "Fundamental Theorem of Calculus."

Corollary 2.3. *Suppose that f is continuous on an open interval I, and let F be any antiderivative of f on I. Then, for each c and each x in I, we have*

$$F(x) = F(c) + \int_c^x f(t)\, dt.$$

This might seem stupid at first, since it's just saying the same thing as the theorems above. However, we want to interpret this in the following way. When we apply this, we are really interested in the value of the antiderivative $F(x)$. This says that you can determine the value of $F(x)$ at a random point precisely by evaluating it at some point $F(c)$ and calculating the integral from x to c. This is useful, for instance, if, say $F(c)$ is easy to calculate for a specific c, but $F(x)$ is generally hard to compute.

3. Applications of the Fundamental Theorem of Calculus

Let's start with some familiar examples.

Example 3. We have

$$\int_0^b x^p\, dx = \frac{b^{p+1}}{p+1}$$

Proof. Note that $f(x) = x^p$ is continuous and bounded on $[0b]$ for any b. Furthermore, we know that

$$\left(\frac{x^{p+1}}{p+1}\right)' = \frac{p+1}{p+1} x^p = x^p$$

for all x, so $\frac{x^{p+1}}{p+1}$ is a primitive of x^p. By the FTC,

$$\int_0^b x^P \, dx = \frac{b^{p+1}}{p+1} - \frac{0}{p+1} = \frac{b^{p+1}}{p+1}$$

as desired. □

This illustrates the power of the FTC to simplify greatly. For instance, as you know from your homework, proving this fact without the FTC takes quite a bit more work. In fact, depending on your approach, it may have been shorter if you just proved the FTC first and then applied it!

The main way that we use the second form of the FTC is to deal with integration of the form

$$F(x) = \int_a^{g(x)} f(t) \, dt$$

where $f(x)$ is some continuous and bounded function. Without the FTC, how can we take the derivative? Taking the derivative of an integral itself it subtle without the FTCs, and to deal with the composition with the function $g(x)$ is difficult.

However, we can attack these with the FTC as follows. Let

$$H(x) = \int_a^x f(t) \, dt,$$

so that $F(x) = H(g(x))$. Then the chain rule says

$$F'(x) = H'(g(x)) \cdot g'(x).$$

By the FTC, we see that $H'(x) = f(x)$, so then have

$$F'(x) = f(g(x)) \cdot g'(x),$$

which is something we can calculate!

Let's see this method in action.

Example 4. Calculate the derivative of the function

$$F(x) = \int_{1/x}^x \frac{1}{t} dt,$$

whenever $t > 0$.

Solution. First, define the function $G(x)$ as

$$G(x) := \int_1^x \frac{1}{t} dt.$$

Then, by the fundamental theorem of calculus, we have that

$$G'(x) := 1/x.$$

So: note that

$$F(x) = \int_{1/x}^x \frac{1}{t} dt = \int_1^x \frac{1}{t} dt - \int_1^{1/x} \frac{1}{t} dt = G(x) - G(1/x).$$

(Note that we defined the function G here as an integral starting at 1, not 0! This is because the integral $\int_0^x \frac{1}{t} dt$ doesn't even exist whenever x is nonzero. So, when you use linearity of your integrals to split them apart, do be careful that you're not accidentally breaking your integral into parts that don't exist!)

Then, with this expression of $F(x) = G(x) - G(1/x)$, we can just proceed by the chain rule:

$$\begin{aligned}
(F(x))' &= (G(x) - G(1/x))' \\
&= G'(x) - (-\frac{1}{x^2}) \cdot G'(1/x) \\
&= 1/x + \frac{1}{x^2} \cdot \frac{1}{1/x} \\
&= 2/x.
\end{aligned}$$

□

4. The interaction between differentiability and integration

So we have sort of a dictionary between the two worlds for nice functions: the "light world" of differentiability, and the "dark world" of integration, with the bridge between them given by the Fundamental Theorem of Calculus.

It's important to understand what actions in one world look like when transported to the other world via these bridges. This should be something you should cultivate as you continue to learn calculus.

Consider two of the main tools we have for differentiation:

- The chain rule, which says that for differentiable f, g, we have $(f(g(x))' = f'(g(x)) \cdot g'(x)$.
- The product rule, which says that for differentiable f, g, we have $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$.

What do these look like in the "dark world" of integration?

Taking the product rule over to the dark side, we obtain the following technique.

Theorem 4.1. *(Integration by Parts) If f, g are a pair of C^1-functions on $[a, b]$, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Of course, you probably learned this as a technique in calculus symbolically as

$$\int u\,dv = uv - \int v\,du.$$

I like to this of this result "physically" as saying that you can switch the function to which you are applying $\frac{d}{dx}$ inside an integral, as long as you switch the sign and add an error term. It's cool to try and do this for some basic mechanics questions, for instance.

Taking the chain rule over to the dark side, we obtain another useful technique.

Theorem 4.2. *(Change of Variables/Substitution) If f is a continuous function on $g([a,b])$ and g is a C^1-function on $[a,b]$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Remark 2. Don't forget to change the endpoints.

Again, you can interpret this as follows. Your $g(x)$ is a sort of "distortion" of the interval, so to account for this, you need to integrate by $g'(x)dx$ instead of just dx itself.

This kind of geometric reasoning may not be so useful in a one-dimensional setting, but you will see that it will be crucial to understand in multivariable calculus, when identifying symmetries and decomposing a function into a composite of simpler functions greatly simplifies calculations.

4.1. **Review of Integration by Parts and Substitution.**

Question 4.3. *What's*

$$\int_1^2 x^2 e^x dx\ ?$$

Solution. Looking at this problem, it doesn't seem like a substitution will be terribly useful: so, let's try to use integration by parts!

How do these kinds of proofs work? Well: what we want to do is look at the quantity we're integrating (in this case, x^2e^x,) and try to divide it into two parts – a "$f(x)$"-part and a "$g'(x)$" part – such that when we apply the relation $\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x)$, our expression gets simpler!

To ensure that our expression does in fact get simpler, we want to select our $f(x)$ and $g'(x)$ such that

1. we can calculate the derivative $f'(x)$ of $f(x)$ and find a primitive $g(x)$ of $g'(x)$, so that either
2. the derivative $f'(x)$ of $f(x)$ is **simpler** than the expression $f(x)$, or
3. the integral $g(x)$ of $g'(x)$ is **simpler** than the expression $g'(x)$.

So: often, this means that you'll want to put quantities like polynomials or $\ln(x)$'s in the $f(x)$ spot, because taking derivatives of these things generally simplifies them. Conversely, things like e^x's or trig functions whose integrals you know are good choices for the integral spot, as they'll not get much more complex and their derivatives are generally no simpler.

Specifically: what should we choose here? Well, the integral of e^x is a particularly easy thing to calculate, as it's just e^x. As well, x^2 becomes much simpler after repeated derivation: consequently, we want to make the choices

$$\begin{array}{ll} f(x) = x^2 & g'(x) = e^x \\ f'(x) = 2x & g(x) = e^x, \end{array}$$

which then gives us that

$$\begin{aligned} \int_1^2 x^2 e^x dx &= f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \\ &= x^2 e^x\Big|_1^2 - \int_1^2 2xe^x dx. \end{aligned}$$

Another integral! Motivated by the same reasons as before, we attack this integral with integration by parts as well, setting

$$\begin{array}{ll} f(x) = 2x & g'(x) = e^x \\ f'(x) = 2 & g(x) = e^x. \end{array}$$

This then tells us that

$$\begin{aligned} \int_1^2 x^2 e^x dx &= x^2 e^x\Big|_1^2 - \int_1^2 2xe^x dx \\ &= x^2 e^x\Big|_1^2 - \left(f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \right) \\ &= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - \int_1^2 2e^x dx \right) \\ &= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - 2e^x\Big|_1^2 \right) \\ &= 4e^2 - e^1 - \left(4e^2 - 2e^1 - 2e^2 + 2e^1\right) \\ &= 2e^2 - e^1. \end{aligned}$$

□

Question 4.4. *What is*

$$\int_0^2 x^2 \sin(x^3) dx \; ?$$

Solution. How do we calculate such an integral? Direct methods seem unpromising, and using trig identities seems completely insane. What happens if we try substitution?

Well: our first question is the following: **what should we pick?** This is the only "hard" part about integration by substitution – making the right choice on what to substitute in. In most cases, what you want to do is to find the part of the integral that you don't know how to deal with – i.e. some sort of "obstruction." Then, try to make a substitution that (1) will remove that obstruction, usually such that (2) the derivative of this substitution is somewhere in your formula.

Here, for example, the term $\sin(x^3)$ is definitely an "obstruction" – we haven't developed any techniques for how to directly integrate such things. So, we make a substitution to make this simpler! In specific: Let $g(x) = x^3$. This turns our term $\sin(x^3)$ into a $\sin(g(x))$, which is much easier to deal with Also, the derivative $g'(x) = 3x^2 dx$ is (up to a constant) being multiplied by our original formula – so this substitution seems quite promising. In fact, if we calculate and use our indicated substitution, we have that

$$\begin{aligned} \int_0^2 x^2 \sin(x^3) dx &= \int_0^2 \sin(g(x)) \cdot \frac{1}{3} \cdot g'(x) dx \\ &= \int_{0^3}^{2^3} \sin(x) dx \\ &= \frac{\cos(0)}{3} - \frac{\cos(8)}{3} \end{aligned}$$

(Note that when we made our substitution, we also changed the bounds from $[a, b]$ to $[g(a), g(b)]$! Please, please, always change your bounds when you make a substitution!) $\square$

5. Why is differentiation easy, but integration so hard?

As you may have noticed, in calculus, integration is much harder than differentiation. But why is this the case, if they are essentially the same, as inverses of each other?

Well, the first thing is that you shouldn't always expect the existence of inverses operations to have similar levels of difficulty. In other words, "inverse" does not always mean "symmetric." For instance, it's easy to mix a needle into a haystack, but it's harder to then separate the needle from the haystack once again. For a slightly more concrete example, think of squaring and taking the square root of rational numbers. The square of a rational number is always rational, but the square root of a rational number may not be!

Another reason that integration is seen as hard is due to the approach that we tend to take when learning calculus: generally speaking, we learn differentiation first, think of it as the "basic" operation, and then thinking of integration as the inverse.

But what if we tried the opposite point of view? (Indeed, Apostol itself follows this approach by defining integration first, but Math 1a generally takes the traditional "differentiation, then integration" approach.)

For instance, consider the following question. Given a function g, when is there an f such that

$$g(x) = C + \int_0^x f(t)\, dt$$

holds? The answer is whenever g is absolutely continuous, which is when for every $\epsilon > 0$, there exists a δ such that whenever we have a finite sequence of pairwise disjoint subintervals (a_k, b_k) such that

$$\sum_k |b_k - a_k| < \delta$$

then

$$\sum_k |f(b_k) - f(a_k)| < \epsilon.$$

This is stronger than continuity, and stronger than a notion called *uniform* continuity (which itself implies continuity). As you may notice, this definition is difficult to check, unless your function is of a specific type.

Every absolutely continuous function is continuous, but there are lots of continuous functions that are not absolutely continuous. Indeed, "most" continuous are not absolutely continuous, in a sense similar to how we saw that "most" continuous functions are not differentiable.

This actually highlights one instance of why differentiation is easy, but integration is hard. For instance, differentiation just relies on "local" information, while integration relies on "global" information. And it's this difficulty in "patching up" local information to get global information that contributes to integration being more difficult.