MA 1: SESSION 7

1. Asymptotic Limits of Functions

So far we have mostly talked about limits of functions at a given point on the real line. However, we can also make rigorous sense of the limit of a function at infinity.

Definition 1.1. For a function $f : \mathbf{R} \to \mathbf{R}$, we say that

$$\lim_{x \to \infty} f(x) = L$$

if for all $\epsilon > 0$, there exists an $N \in \mathbf{R}$ such that if $x > N$, then $|f(x) - L| < \epsilon$.

Remark 1. While the symbols looks formally similar to limits of *sequences* as $n \to \infty$, e.g. $\lim_{n \to \infty} \frac{1}{n} = 0$, the meanings are different.

For example, now we can make sense of statements like

$$\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 1}{3x^3 - 4} = \frac{4}{3}$$

that are familiar from your previous calculus classes.

Example 1. We have

$$\lim_{x \to \infty} \frac{1}{x^2} = 0.$$

Proof. Let $\epsilon > 0$. Set $N = \max\{1, \frac{1}{\sqrt{\epsilon}}\}$. For all $x > N$, we have

$$|f(x) - L| = \left| \frac{1}{x^2} \right| < \frac{1}{N^2} \leq \frac{1}{1/\sqrt{\epsilon}^2} = \epsilon.$$

$\square$

Note that we needed N to be at least 1, to ensure that $x > N$ implies that $x^2 > N^2$.

We can also make sense of when the limit of a function is infinite.

Definition 1.2. We say that

$$\lim_{x \to \infty} f(x) = \infty$$

if for every positive N there is a number M such that if $x > M$, then $f(x) > N$.

Example 2. We have

$$\lim_{x \to \infty} x^2 = \infty.$$

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Proof. Let $N > 0$. Set $M = \max\{1, N\}$. Then for all $x > M$, we have

$$x^2 > M^2 \geq N.$$

□

We needed to ensure that $M \geq 1$ so that $M^2 \geq M$.

A function not having a limit and a function having a limit that is ∞ are different notions. This is usually not emphasized in your first calculus course, since you need something like ϵ-δ proofs to make the notion rigorous. However, do not confuse these two notions.

In summary, we have three possibilities for the limit of a function as $x \to \infty$:

(1) The limit exists, and is finite.
(2) The limit exists, and is infinite.
(3) The limit does not exist.

All three classes are distinct.

We've seen examples of the first two types. Let's give an example of the third type.

Example 3. The limit of $\cos(x)$ as $x \to \infty$ does not exist.

Proof. Since $|\cos(x)| \leq 1$, its limit as $x \to \infty$ is not ∞. It remains to show that $\cos(x)$ does not have any finite limit as $x \to \infty$.

Suppose for contradiction that

$$\lim_{x\to\infty} \cos(x) = L$$

where $L \in \mathbf{R}$. We must show that for some $\epsilon > 0$ that there does not exist any $N \in \mathbf{R}$ such that $x > N$ forces $|f(x) - L| < \epsilon$.

Note that $\cos(x)$ is 2π-periodic, that is, $\cos(x) = \cos(x+2\pi)$ for all $x \in \mathbf{R}$, and so for any N, we have an equality of sets

$$\{\cos(x) \mid x > N\} = \{\cos(x) \mid x \in \mathbf{R}\}.$$

Suppose that $|L| > 1$. Consider the value

$$|\cos(x) - L| =: C_x > 0.$$

Note that $C_x > 0$ for all x since $|\cos(x)| \leq 1$. Set $C = \min_x C_x$. Then for any $\epsilon > 0$ such that $C > \epsilon$, there does not exist any N such that $x > N$ forces $|f(x) - L| < \epsilon$.

Suppose that $|L| \leq 1$. We claim that $\epsilon = \frac{1}{2}$ fails our desired condition.

Say that $L \geq 0$. For any N, there exists an $x > N$ such that $\cos(x) = -1$ and so

$$|\cos(x) - L| = L + 1 \geq \epsilon = \frac{1}{2}.$$

Say that $L \leq 0$. For any N, there exists an $x > N$ such that $\cos(x) = 1$ and so

$$|\cos(x) - L| = |L| + 1 \geq \epsilon = \frac{1}{2}.$$

□

2. One-Sided Limits

Let's conclude with something fairly elementary: the concept of a **one-sided limit**.

Definition 2.1. For a function $f : X \to Y$, we say that

$$\lim_{x \to a^+} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to a^-} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x < a) \Rightarrow (|f(x) - L| < \epsilon).$$

Basically, this is just our original definition of a limit except we're only looking at x-values on one side of the limit point a: hence the name "one-sided limit." Thus, our methods for calculating these limits are pretty much identical to the methods we introduced before: we work one example below, just to reinforce what we're doing here.

Proposition 2.2.

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Proof. First, examine the quantity

$$\frac{|x|}{x}.$$

For $x > 0$, we have that

$$\frac{|x|}{x} = 1;$$

therefore, for any $\epsilon > 0$, it doesn't even matter what δ we pick! – because for any x with $0 < x$, we have that

$$\left| \frac{|x|}{x} - 1 \right| = 0 < \epsilon.$$

Thus, the limit as $\frac{|x|}{x}$ approaches 0 from the right hand side is 1, as desired. □

Remark 2. Note that this is one of the rare instances where $\delta > 0$ does not depend on ϵ.

One-sided limits are particularly useful when we're discussing limits at infinity, as we describe in the next section.

2.1. **Limits at Infinity as One-Sided Limits.**

Definition 2.3. For a function $f : X \to Y$, we say that

$$\lim_{x \to +\infty} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to -\infty} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x < N) \Rightarrow (|f(x) - L| < \epsilon).$$

In class, we described a rather useful trick for calculating limits at infinity:

Proposition 2.4. *For any function $f : X \to Y$,*

$$\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right).$$

Similarly,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right).$$

The use of this theorem is that it translates limits at infinity (which can be somewhat complex to examine) into limits at 0, which can be in some sense a lot easier to deal with: as opposed to worrying about what a function does at extremely large values, we can just consider what a different function does at rather small values (which can make our lives often a lot easier.)

3. Kinds of Infinity

The material presented seems to not be known by everybody, but is important for placing the results of this class in the context of your greater scientific understanding.

Let's start with one of the basic questions from the beginning of the course. What do we mean when we say that a set is infinite?

To talk about this notion, let's start by recalling a definition.

Definition 3.1. We say that a set S is **countable** if it is finite or there is a map of sets $f : \mathbf{N} \to S$ such that $f(x) \neq f(y)$ for $x \neq y$.

We say that a set S is **countably infinite** if S is countable but not finite.

For example, $\mathbf{N}$ itself is countably infinite. The integers **ZZ** are countably infinite, we can list them as follows:

$$0, +1, -1, +2, -2, +3, -3, \ldots.$$

The prime numbers $\mathbf{N}$ are also countably infinite.

The set of all possible words in a given alphabet is countably infinite. (Of course, many of these "words" are nonsense, and there are only finitely many English words in usage...the dictionary is certainly not infinitely long.)

The most striking example was one that was covered in class.

Proposition 3.2. *The rational numbers are countably infinite.*

Proof. We can arrange the rational numbers in the following way

$$0/1, 1/1, -1/1, 2/1, -2/1, 1/2, -1/2, 1/3, -1/3, 3/1, -3/1, 4/1, -4/1, \ldots.$$

Every rational number will appear somewhere on this list. Since it is possible to map each rational number to its position on this list (that is, an element of $\mathbf{N}$), the set of rational numbers is countable. □

Now, what about the real numbers? Are they countably infinite?

The answer is no, and this is one of the most famous proofs in mathematics, due to Cantor. Recall how the argument went.

Proposition 3.3. *The real numbers are not countable.*

Proof. Let S be any countable list of real numbers. We will show that there exists an element $r \in \mathbf{R}$ that is not contained in S.

Let s_i be the ith element of the list S. Define r so that its ith digit is different from the ith digit of s_i. Then by definition, $r \notin S$. □

Corollary 3.4. *The irrational numbers* $\boldsymbol{R} - \boldsymbol{Q}$ *are not countable.*

Even though we now know, through the homework assignments, that the $\mathbf{Q}$ are ever-present in $\mathbf{R}$—between any two distinct real numbers, there is a rational number—"most" of $\mathbf{R}$ consists of the irrationals.

Of course, once we introduce this level of largeness, we also have some more weirdness.

Proposition 3.5. *The number of elements in* $\boldsymbol{R}$ *is the same as the number of element in any open interval* (a, b)*. We have a bijection*

$$\begin{aligned} f : \boldsymbol{R} &\to (a, b) \\ x &\mapsto \frac{\arctan x + \frac{\pi}{2}}{\pi}(b - a) + a. \end{aligned}$$

And the existence of "space-filling curves" gives the following fact.

Proposition 3.6. *We* $|\boldsymbol{R}^n| = |\boldsymbol{R}|$ *for any* n*.*

We have something even more bizarre, though. Let's consider the following kind of number.

Definition 3.7. An **algebraic number** a is a number that is the root of a polynomial with coefficients in $\mathbf{Q}$.

All rational numbers are algebraic, since any $a \in \mathbf{Q}$ is the root of the polynomial

$$x - a$$

which has rational coefficients.

However, the algebraic numbers contain much more: for example, $\sqrt{2}$ is an irrational number, but it's algebraic, because it's the root of

$$x^2 - 2.$$

Similarly, every nth root is an algebraic number. The algebraic numbers have nice properties: the sum of two algebraic numbers is algebraic, and the product of two algebraic numbers is also algebraic. (Try and work these out!)

Roughly speaking, the algebraic numbers represent the real numbers you get by starting with the integers and adding, subtracting, multiplying, dividing, and taking roots.

This leads us to ask, are there any real numbers that are not algebraic, for which there does not exist *any* polynomial, however large, for which it is a root?

Proposition 3.8. *There are countably many algebraic numbers.*

Proof. The list of all polynomials with rational coefficients is countable. (We can, say, order by degree.) □

Corollary 3.9. *There are uncountably many numbers are that are not algebraic. These are called* ***transcendental*** *numbers.*

Thus, in a very concrete sense, *most* real numbers are not even algebraic.

What is an example of a transcendental number? What is a number that is not the root of *any* polynomial with rational coefficients?

Example 4. The first explicit example is attributed to Liouville, it's the number

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.11000100000000\ldots.$$

It's fun to try and undersatnd why the number above cannot satisfy any possible polynomial equation (one way is to think about the growth of the sequence and how to account for it as a polynomial.

Some familiar constants are also known to be transcendental. It's a little hard to prove, but if you were to look it up, you could probably understand it, using the tools we've developed in this class.

Theorem 3.10. *The exponential e is transcendental.*

Theorem 3.11. *π is transcendental.*

These results have been known for a long time. However, it's still an open question whether things like $\pi + e$ is transcendental.

Thus, despite the fact that almost all numbers are transcendental, all but a couple of the numbers that we've every encountered is algebraic.

What's really amazing is that the somewhat crude definition that we use to define real numbers— essentially, numbers whose decimal expansions can possibly extend infinitely to the left and right of the decimal point—is a sufficient setting in which we can do calculus and do mathematics that actually models things in the real world.