MA 1: SESSION 8

1. Spot the Mistake

Consider the following answer to a problem from the homework.

Example 1. $\lim_{x\to\infty} x(\sqrt{x+2}-\sqrt{x}) = 0$

"Proof". We first show that

$$\lim_{x\to\infty} \sqrt{x+2}-\sqrt{x} = 0.$$

Let $\epsilon > 0$. Set $M = \max(1, \frac{4}{\epsilon^2})$. Then for $x > M$, we have

$$\begin{aligned}
|\sqrt{x+2}-\sqrt{x}| &= \left|\frac{(\sqrt{x+2}-\sqrt{x})(\sqrt{x+2}+\sqrt{x})}{\sqrt{x+2}+\sqrt{x}}\right| \\
&= \left|\frac{x+2-x}{\sqrt{x+2}+\sqrt{x}}\right| = \frac{2}{|\sqrt{x+2}+\sqrt{x}|} \\
&< \frac{2}{\sqrt{x}} \\
&\leq \frac{2}{\sqrt{M}} \\
&\leq \frac{2}{\sqrt{4/\epsilon^2}} \\
&= \epsilon.
\end{aligned}$$

We then show that $\lim_{x\to\infty} x = \infty$. Let $N > 0$. Set $M = N$. Then for all $x > M$, we have $x > M = N$, as desired.

Finally, since the limits of the terms in the product exist, we have

$$\lim_{x\to\infty} x(\sqrt{x+2}-\sqrt{x}) = \lim_{x\to\infty} x \lim_{x\to\infty} (\sqrt{x+2}-\sqrt{x}) = \infty \cdot 0 = 0.$$

$\square$

Where does it go wrong?

The problem is with the last part. Limits don't commute this way as $x \to \infty$. Again, please be careful when you make these kinds of "obvious" steps, and if in doubt, check the textbook version of the theorem.

Date: November 19, 2015.

2. Taylor Series

2.1. **Definitions.** When we first defined the derivative, recall that it was supposed to be the "instantaneous rate of change" of a function $f(x)$ at a given point a. In other words, f' gives us a linear approximation to $f(x)$ near a: for small ε, we have $f(a+\varepsilon) \approx f(a) + \varepsilon f'(a)$.

Taylor series is just the extension of this idea to higher order derivatives, giving us a better approximation to $f(x)$.

Definition 2.1. Let $f(x)$ be n-times continuously differentiable on an interval $[a,b]$ and let $c \in (a,b)$. Then, the **n-th order Taylor polynomial of $f(x)$ about c** is:

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

The associated **n-th order remainder of** $f(x)$ is defined to be

$$R_n(f)(x) = f(x) - T_n(f)(x)$$

The following two theorems are the essence of why Taylor series is indeed a good approximation of $f(x)$.

Theorem 2.2. *$R_n(f)(x)$ is $o((x-c)^n)$.*

Note that this really just means that when x is close to c, $T_n(f)(x)$ is close to the real value of $f(x)$.

Theorem 2.3. *Suppose $f(x)$ is $(n+1)$-times continuously differentiable. Then,*

$$R_n(f)(x) = \int_c^x \frac{f^{(n+1)}(c)}{n!}(x-y)^n dy$$

If instead of stopping at n, we consider infinitely many terms, we call this a **Taylor series** or **Taylor expansion**.

Example 2. Let's compute the Taylor series for $f(x) = e^x$ about 0.

For all n, we know that $f^{(n)}(x) = e^x$. Hence, we get

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Note also that the radius of convergence for this series is ∞.

Example 3. Consider $f(x) = \frac{1}{x}$. We can't expand this Taylor series at 0, because f is not defined there, so let's try and expand about 1 instead. We first compute

$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = 2\frac{1}{x^3}$$
$$f'''(x) = -6\frac{1}{x^4}$$

so the Taylor expansion of f about 1 is

$$\begin{aligned} f(x) &= 1 - 1(x-1) + \frac{2}{2!}(x-1)^2 - \frac{3\cdot 2}{3!}(x-1)^3 + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \cdots . \end{aligned}$$

Let's see an example of it used as an approximation.

Example 4. We want to determine the "error" of the 5th-degree Taylor approximation for $f(x) = \cos(x)$ centered around π, in other words, a bound on the difference $|f(x) - T_5(x)|$. By the remainder formula, the error is bounded by

$$R_5 = \frac{f^{(6)}(c)}{6!}(x-\pi)^6$$

for some $c \in (a, x)$, the maximizer of this quantity. We have $f^{(6)}(x) = -\sin(x)$ and so we bound the error:

$$|f(x) - T_5(x)| \leq |R_5| = \left| \frac{f^{(6)}(c)}{6!}(x-\pi)^6 \right| \leq \frac{1}{6!}(x-\pi)^6$$

because $|-\sin(x)| \leq 1$. So the error at $x = 3$ is bounded by $\frac{1}{720}(\pi - 3)^6 < 1.11 \times 10^{-9}$. In contrast, the error at $x = 1$ is bounded by $\frac{1}{720}(\pi - 1)^6 < 0.003$, which is much worse.

Obviously, a function needs to be (infinitely) differentiable to have a Taylor expansion. But, surprisingly enough, not all infinitely differentiable functions have a Taylor expansion.

Remark 1. We won't need this for this class, but a function that equals its Taylor expansion is called an **analytic** function.

The classical example of something that is infinitely differentiable but non-analytic is the following function.

Example 5. Consider the function

$$f(x) = \begin{cases} \exp(-1/x), & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

It is easy to see that this function has continuous derivatives of all orders on the entire real line.

But its derivatives at the origin are 0, so its Taylor series does not equal $f(x)$ for $x > 0$. In other words, f is not analytic at 0.

However, most of the functions that we'll consider in this class: e.g. trigonometric functions, logarithms, exponentials, polynomials, are all analytic.

2.2. **Approximating Integrals.** Now, we will see how Taylor series can help us approximate integrals. For example, consider the Gaussian integral $\int e^{-x^2}dx$. Unfortunately, there is no elementary antiderivative of e^{-x^2}. But using Taylor series, we can approximate the value of this integral.

Example 6. Approximate $\int_0^{1/3} e^{-x^2}dx$ to within 10^{-6} of its actual value.

From now, instead of writing $T_n(e^{-x^2})(x)$ and $R_n(e^{-x^2})(x)$, we will simply write $T_n(x)$ and $R_n(x)$. Then, for any n, we have $e^{-x^2} = T_n(x) + R_n(x)$. Thus,

$$\int_0^{1/3} e^{-x^2}dx = \int_0^{1/3} T_n(x)dx + \int_0^{1/3} R_n(x)dx$$

Note that $T_n(x)$ is just a polynomial. Therefore, $\int_0^{1/3} T_n(x)dx$ is an integral that we can explicitly compute. On the other hand, we know that $R_n(x)$ goes to 0 as n increases. So the idea is to make $|\int R_n dx|$ small by increasing n: in this case, we want to find n such that $|\int_0^{1/3} R_n(x)dx| < 10^{-6}$.

By Taylor's Theorem, we have

$$R_n(e^{-x})(x) = \int_0^x (-1)^n \frac{e^{-t}}{n!}(x-t)^n dt$$

Note that $R_n(x)$ from above is $R_n(e^{-x^2})(x)$. However, $R_n(e^{-x^2})(x)$ is precisely equal to $R_n(e^{-x})(x^2)$. Hence, we see that

$$R_n(x) = \int_0^{x^2} (-1)^n \frac{e^{-t}}{n!}(x^2-t)^n dt$$

Unfortunately this is not something we can easily integrate. However, we are not interested in the actual value of this integral. We only need to bound it by 10^{-6}.

First, we see that on the interval $[0, x^2]$, e^{-t} is always less than or equal to $e^0 = 1$. And $(x^2 - t)^n$ is bounded from above by $(x^2 - 0)^n = x^{2n}$. Note also that for all $t \in [0, x^2]$, $\frac{e^{-t}}{n!}(x^2-t)^n \geq 0$. Therefore, we get

$$\begin{aligned}
|R_n(x)| &= \left|\int_0^{x^2} (-1)^n \frac{e^{-t}}{n!}(x^2-t)^n dt\right| \\
&= \textstyle\int_0^{x^2} \frac{e^{-t}}{n!}(x^2-t)^n dt \\
&\leq \textstyle\int_0^{x^2} \frac{1}{n!}x^{2n} dt \\
&= \frac{x^{2n}}{n!}t\Big|_0^{x^2} \\
&= \frac{x^{2n+2}}{n!}
\end{aligned}$$

In our case, we want $|\int_0^{1/3} R_n(t)dt| < 10^{-6}$. Therefore, we need to find a value of n for which $\frac{1}{n!}\left(\frac{1}{3}\right)^{2n+2} < 10^{-6}$. A little playing around with the inequality will tell you that when we let $n = 3$, the inequality is satisfied.

This means that $\int_0^{1/3} T_3(x)dx$ is within 10^{-6} of the real value of $\int_0^{1/3} e^{-x^2}dx$. Again, $\int_0^{1/3} T_3(x)dx$ is very easy to compute explicitly:

$$\begin{aligned}
\int_0^{\frac{1}{3}} T_3(x)dx &= \int_0^{\frac{1}{3}} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right)\Bigg|_0^{\frac{1}{3}} \\
&= \frac{147604}{459270}
\end{aligned}$$

This example shows that Taylor series can be used efficiently to approximate integrals. However, we should note that Taylor series works well *near* the point at which we are writing the series. For example, if we were to follow the exact same steps in the above example to approximate $\int_0^2 e^{-x^2}dx$ to within 0.1 of the actual value, we would need to compute $\int_0^2 T_{11}(x)dx$. In our example, we only needed third order Taylor expansion to get an approximate value with error less than 10^{-6}. However, as we get farther away from 0, to get an approximate with similar error bounds, we need to use higher and higher order Taylor polynomials.

One way to fix this is to divide the interval into several subintervals. Then, we can write a Taylor series expansion for $f(x)$ for each interval and approximate the integral over each interval as we did in above example.

3. Proving facts about functions in an open neighborhood.

Example 7. Suppose f is a differentiable function on $\mathbf{R}$ with bounded derivative (say $|f'| \leq M$). Fix an $\epsilon > 0$ and define

$$g(x) = x + \epsilon f(x).$$

Prove that g is one-to-one if ϵ is small enough.

Proof. We observe that if a function $p(x)$ is has positive derivative in an interval (a, b), then $p(x)$ is strictly increasing on (a, b).

Claim. Any strictly increasing function is one-to-one.

Suppose that $f(x) = f(y)$. Suppose for contradiction that $x \neq y$, we can assume without loss of generality that $x > y$. But since f is strictly increasing, we have $f(x) > f(y)$, a contradiction.

When $0 < \epsilon < 1/M$, we have

$$f'(x) = 1 + \epsilon g'(x) \geq 1 - \epsilon M > 0.$$

Thus, f is strictly increasing and thus one-to-one. □

Question 3.1. *Let f be a continuous real function on $\boldsymbol{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?*

Answer. Yes. By the Mean Value Theorem or L'Hopital's rule, we see that
$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}f'(x)$$
exists if the last limit exists, no matter if it's 3 or not. $\square$

Theorem 3.2. *(L'Hôpital's Rule) Assume that f and g have derivatives $f'(x)$ and $g'(x)$ at each point x of an open interval (a,b), and suppose that*
$$\lim_{x\to a+}f(x)=0 \quad \text{and} \quad \lim_{x\to a+}g(x)=0.$$
Assume also that $g'(x)\neq 0$ for each x in (a,b). Then if
$$\lim_{x\to a+}\frac{f'(x)}{g'(x)}=L,$$
it implies that
$$\lim_{x\to a+}\frac{f(x)}{g(x)}=L.$$

Example 8. Suppose that f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that
$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x).$$
Show by an example that the limit may exist even if $f''(x)$ does not.

Solution. Since $f''(x)$ exists, f is continuous near x and so
$$\lim_{h\to 0}f(x+h)+f(x-h)-2f(x)=0.$$
By l'Hôpital's rule,
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}&=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\&=\lim_{h\to 0}\left(\frac{f'(x+h)-f'(x)}{2h}+\frac{f'(x-h)-f'(x)}{2(-h)}\right)\\&=\frac{f''(x)}{2}+\frac{f''(x)}{2}=f''(x).\end{aligned}$$
As for the example, $f(x)=x|x|$ works. $\square$