MA 1: SESSION 9

1. Final Review

1.1. Fundamental Theorem of Calculus.

Question 1.1. *Suppose that f is a function such that $f''(x) = -3$, and $f'(1) = f(1) = 0$. What is f?*

Proof. By the Fundamental Theorem of Calculus, we know that (because f' is a primitive of f'')

$$\begin{aligned} f'(x) &= f'(1) + \int_1^x f''(t)\,dt \\ &= 0 + \int_1^x -3\,dt \\ &= -3x + 3. \end{aligned}$$

Applying the theorem again (since f is a primitive of f'), we obtain

$$\begin{aligned} f(x) &= f(1) + \int_1^x f'(t)\,dt \\ &= 0 + \int_1^x -3x + 3\,dt \\ &= -\frac{3x^2}{2} + 3x - \frac{3}{2} \\ &= -\frac{3}{2}(x-1)^2. \end{aligned}$$

□

Check: It's good to check that your answer makes sense. Let's try to check this answer another way, by trying to graph f using the data given.

Since the second derivative of f is always negative, the first derivative of f is always decreasing and also continuous (by the continuity of the integral and realizing that $(f')' = f''$). Since $f'(1) = 0$, when then know that $f'(x)$ is only equal to 0 at 1 (since it's always decreasing), and so f has its only critical point at 0.

Since $f'' < 0$, this point is a maximum and the graph is concave-down everywhere. Then using the fact that $f(1) = 0$ tells us that the graph looks like a downward-opening parabola from $(1, 0)$, which agrees with our answer.

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1.2. ϵ-δ **proofs and complex polar coördinates.**

Question 1.2.

(1) *Prove that* $f(x) = x^3 - 1$ *is a continuous function on all of* $\mathbb{R}$.
(2) *What are this function's roots over* $\mathbb{C}$?
(3) *What are this function's global minima and maxima over the interval* $[-1, 1]$?

Proof. (1): To prove this, let's follow the recipe laid out previously. Pick $a \in \mathbf{R}$. (Since we've done a number of examples where I just wrote the proof and only one where I went through the process, let me go through the thought process again.)

(1) First, let's look at $|f(x) - f(a)|$, and try to create a simple bound depending only on $|x - a|$ and some constants.

$$|f(x) - f(a)| = |x^3 - 1 - a^3 + 1| = |x^3 - a^3| = |x - a| \cdot |x^2 + xa + a^2|.$$

If x is within, say, 1 of a, we know that we can bound this quantity $|x^2 + xa + a^2|$ as follows:

$$|x^2 + xa + a^2| \leq |(a + 1)^2 + a(a + 1) + a^2| \leq 3(a + 1)^2,$$

which is a constant! Therefore, whenever x is within 1 of a, we have the following simple bound:

$$|f(x) - f(a)| \leq |x - a| \cdot (3(a + 1)^2).$$

(2) Now that we have this nice constant bound, we want to pick δ such that whenever $|x - a| < \delta$, $|f(x) - f(a)| < \epsilon$. To do this, we simply want to pick δ such that
   - $\delta < 1$, so that x is always forced to be within 1 of a, and we have our nice constant bound, and
   - $\delta < \frac{\epsilon}{3(a+1)^2}$, because this means that

$$|f(x) - f(a)| \leq |x - a| \cdot (3(a + 1)^2) < \frac{\epsilon}{3(a + 1)^2} \cdot 3(a + 1)^2 = \epsilon$$

So: let $\delta < \min\left(1, \frac{\epsilon}{3(a+1)^2}\right)$.

Then δ is smaller than both 1 and $\frac{\epsilon}{3(a+1)^2}$, and so both of our above statements hold! In particular, for any epsilon, this choice of δ forces

$$|f(x) - f(a) < \epsilon,$$

which is exactly what we want to do in an $\epsilon - \delta$ proof to show continuity.

(2): Finding this function's roots over $\mathbb{C}$ is equivalent to finding all of the values of z such that

$$1 = z^3.$$

To do this: first, remember that we can write any nonzero point in $\mathbb{C}$ with polar coördinates (r, θ) uniquely in the form $re^{i\theta}$, where $r \in (0, \infty)$ and $\theta \in [0, 2\pi]$. Then, we're just looking for all of the values r, θ such that

$$1 = r^3 e^{3i\theta}.$$

Notice that if the above equation holds, then we have that

$$1 = \left| r^3 e^{3i\theta} \right| = \left| r^3 \right| \cdot \left| e^{3i\theta} \right|.$$

However, if we use the formula $e^{ix} = \cos(x) + i\sin(x)$ and the definition $|a + bi| = \sqrt{a^2 + b^2}$, we can see that

$$\begin{aligned} \left| e^{3i\theta} \right| &= |\cos(3\theta) + i\sin(3\theta)) \\ &= \sqrt{\cos^2(3\theta) + \sin^2(3\theta)} \\ &= \sqrt{1} \\ &= 1. \end{aligned}$$

Therefore, we in fact have that $r^3 = 1$; i.e. $r = 1$! All we have to do now is then solve for θ.

We do this in a similar way: if we have $e^{3i\theta} = 1$, by using $e^{ix} = \cos(x) + i\sin(x)$ again, we must have that

$$\begin{aligned} & 1 = \cos(3\theta) + i\sin(3\theta) \\ \Rightarrow \qquad & \cos(3\theta) = 1, \text{ and } \sin(3\theta) = 0. \end{aligned}$$

The three values $\theta = 0, 2\pi/3, 4\pi/3$ are solutions to the above, and therefore correspond to the three roots $1, e^{2i\pi/3}, e^{4i\pi/3}$ of $f(z) = z^3 - 1$; by the fundamental theorem of algebra, we know that there are only three roots, and thus that we've found them all.

(3): Finally, we can find the minima and maxima of this (now real-valued, again) function on $[-1, 1]$ by simply taking its derivative. As $f'(x) = 3x^2$ has its only 0 at 0, we know (by the extremal value theorem) that the only points we have to check for extrema are $x = -1, 0$, and 1. Because $f(-1) = -2, f(0) = -1$, and $f(1) = 0$, we know that its global maxima on this interval is 0 and its global minima is -2. $\square$

1.3. **L'Hôpital's rule / functions of the form** $(f(x))^{g(x)}$ **:**

Question 1.3. *Show that the limit*

$$\lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3}$$

converges to $1/2$.

Proof. We will have to use L'Hôpital's rule multiple times to evaluate this limit. First, before we can apply L'Hôpital's rule, we must check that its conditions apply. The functions contained in the numerator and denominator are all infinitely differentiable near 0, so this will never be a stumbling block: furthermore, because the numerator and denominator are both continuous/defined at 0, we can evaluate their limits at 0 by just plugging in 0: i.e.

$$\lim_{x\to 0}(1-x)^x - 1 + x^2 = (1-0)^0 - 1 + 0^2 = 1 - 1 = 0, \text{ and}$$
$$\lim_{x\to 0} x^3 = 0^3 = 0.$$

So we've satisfied the conditions for L'Hôpital's rule, and can apply it to our limit:

$$\lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3} =_{L'H} \lim_{x\to 0} \frac{\frac{d}{dx}\left((1-x)^x - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)}.$$

At this point, we recall how to differentiate functions of the form $f(x)^{g(x)}$, where $f(x) > 0$, by using the identity

$$\begin{aligned} (f(x))^{g(x)} &= e^{\ln(f(x))\cdot g(x)} \\ \Rightarrow \frac{d}{dx}(f(x))^{g(x)} &= \frac{d}{dx} e^{\ln(f(x))\cdot g(x)} \\ &= e^{\ln(f(x))\cdot g(x)} \cdot \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x)\ln(f(x))\right). \end{aligned}$$

In particular, we can rewrite $(1-x)^x$ as $e^{\ln(1-x)\cdot x}$, which will let us just differentiate using the chain rule:

$$\begin{aligned} \lim_{x\to 0} \frac{(1-x)^x - 1 + x^2}{x^3} &=_{L'H} \lim_{x\to 0} \frac{\frac{d}{dx}\left((1-x)^x - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)} \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} - 1 + x^2\right)}{\frac{d}{dx}\left(x^3\right)} \qquad = \lim_{x\to 0} \frac{e^{\ln(1-x)\cdot x}\cdot\left(\ln(1-x) + \frac{x}{x-1}\right) + 2x}{3x^2} \end{aligned}$$

Again, both the numerator and denominator are continuous, and plugging in 0 up top yields $e^{ln(1)\cdot 0} \cdot \left(\ln(1) + \frac{0}{1}\right) - 2\cdot 0 = 0$, while on the bottom we also get 0. Therefore, we can apply L'Hôpital's rule again to get that our limit is just

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right) + 2x\right)}{\frac{d}{dx}\left(3x^2\right)}$$
$$= \lim_{x\to 0} \frac{e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^2 + e^{\ln(1-x)\cdot x} \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) + 2}{6x}$$

Again, the top and bottom are continuous near 0, and at 0 the top is

$$e^{\ln(1-0)\cdot 0} \cdot \left(\ln(1-0) + \frac{0}{0-1}\right)^2 + e^{\ln(1-0)\cdot 0} \cdot \left(-\frac{1}{1-0} - \frac{1}{(0-1)^2}\right) + 2 = 0 - 2 + 2 = 0,$$

while the bottom is also 0. So, we can apply L'Hôpital again! This tells us that our limit is in fact

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^2 + e^{\ln(1-x)\cdot x} \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) + 2\right)}{\frac{d}{dx}\left(6x\right)}$$
$$= \lim_{x\to 0} \frac{\begin{array}{r} e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right)^3 \quad +3e^{\ln(1-x)\cdot x} \cdot \left(\ln(1-x) + \frac{x}{x-1}\right) \cdot \left(-\frac{1}{1-x} - \frac{1}{(x-1)^2}\right) \\ +e^{\ln(1-x)\cdot x} \cdot \left(\frac{-1}{(x-1)^2} + \frac{2}{(x-1)^3}\right) \end{array}}{6}.$$

Again, the top and bottom are made out of things that are continuous at 0. Plugging in 0 to the top *this time* gives us -3, while the bottom gives us 6: therefore, the limit is just

$$\frac{-3}{6} = -\frac{1}{2}.$$

So we're done! □

1.4. **Taylor series:**

Question 1.4. *Approximate the integral*

$$\int_1^2 \frac{\sin(x)}{x} dx.$$

Proof. Recall that the Taylor series for $\sin(x)$:

$$T(\sin(x)) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Using this, we can write

$$\frac{\sin(x)}{x} = \frac{T_n(\sin(x))}{x} + \frac{R_n(\sin(x))}{x},$$

and therefore write

$$\int_1^2 \frac{\sin(x)}{x}dx = \int_1^2 \frac{T_n(\sin(x))}{x}dx + \int_1^2 \frac{R_n(\sin(x))}{x}dx.$$

We do this for the same reasons as in our estimation of the Gaussian integral $\int e^{-x^2}$! Specifically, notice that the $\frac{T_n(\sin(x))}{x}$ part is just a polynomial:

$$\frac{T_n(sin(x))}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

which should be easy to integrate.

This leaves the $\frac{R_n(\sin(x))}{x}$ part, which we should be able to bound using Taylor's theorem. Specifically, we have

$$\begin{aligned} R_n(\sin(x)) &= \int_0^x \frac{\frac{d^n}{dt^n}(\sin(t))}{n!} \cdot (x-t)^n dt \\ \Rightarrow |R_n(\sin(x))| &\leq \int_0^x \frac{\left|\frac{d^n}{dt^n}(\sin(t))\right|}{n!} \cdot |(x-t)^n|\, dt \\ &\leq \int_0^x \frac{1}{n!} \cdot x^n dt \\ &= \left(\frac{x^n}{n!}\right) \cdot t\Big|_0^x \\ &= \frac{x^{n+1}}{n!}. \end{aligned}$$

Therefore, we can bound the integral $\int_1^2 \frac{R_n(\sin(x))}{x}$ as follows:

$$\left|\int_1^2 \frac{R_n(\sin(x))}{x}dx\right| \leq \int_0^2 \frac{x^n}{n!}dx = \frac{x^{n+1}}{(n+1)!}\Big|_1^2 = \frac{2^{n+1}-1}{(n+1)!}.$$

This quantity is $\frac{7}{80} < .1$ at $n = 6$. Therefore, we've proven that

$$\int_1^2 \frac{\sin(x)}{x}dx = \int_1^2 \frac{T_6(\sin(x))}{x}dx,$$

up to $\pm .1$.

So: to find this integral, it suffices to integrate $\frac{T_6(\sin(x))}{x}$. This is trivial:

$$\begin{aligned}\int_1^2 \frac{T_6(\sin(x))}{x}dx &= \int_1^2 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{x}dx \\ &= \int_1^2 1 - \frac{x^2}{3!} + \frac{x^4}{5!}dx \\ &= \left(x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!}\right)\Big|_1^2 \\ &= \frac{1193}{1800} \\ &\approx .66\end{aligned}$$

So our integral is about $.66 \pm .1$.

$\square$

1.5. **Integration methods:**

Question 1.5. *Calculate the following two integrals:*

$$\int_0^1 \ln(1+x^2)dx, \quad \int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx.$$

Proof. We begin by studying $\int_0^1 \ln(1+x^2)dx$. Because no substitution looks very promising (as the $1+x^2$ term messes things up,) we are motivated to try integration by parts. In particular, we can remember the trick we used when integrating $\ln(x)$, and set

$$\begin{array}{ll} u = \ln(1+x^2) & dv = dx \\ du = \frac{2x}{1+x^2} & v = x, \end{array}$$

which gives us

$$\int_0^1 \ln(1+x^2)dx = \ln(1+x^2)\cdot x\Big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2}dx$$

A bit of algebra allows us to notice that

$$\begin{aligned}\ln(1+x^2)\cdot x\Big|_0^1 - 2\int_0^1 \frac{x^2}{1+x^2}dx &= \ln(1+x^2)\cdot x\Big|_0^1 - 2\left(\int_0^1 1 - \frac{1}{1+x^2}dx\right) \\ &= \ln(1+x^2)\cdot x\Big|_0^1 - 2x\Big|_0^1 + 2\int_0^1 \frac{1}{1+x^2}dx.\end{aligned}$$

Now, we remember our inverse trig identities, and specifically remember that $\int \frac{1}{1+x^2}dx = \arctan(x)$; combining, we have

$$\begin{aligned}\int_0^1 \ln(1+x^2)dx &= \ln(1+x^2)\cdot x\Big|_0^1 - 2x\Big|_0^1 + 2\arctan(x)\Big|_0^1 \\ &= \ln(2) - 2 + \frac{\pi}{2}.\end{aligned}$$

We now look at $\int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx$. Before we can do anything, we have to do some algebra to clean up this function. Specifically, to simplify this expression, we multiply top and bottom by $\sqrt{x+1}-\sqrt{x-1}$, a common algebraic technique used on square-root-involving expressions to clean things up:

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx &= \int_2^3 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx \\ &= \int_2^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{(\sqrt{x+1})^2-(\sqrt{x-1})^2}dx \\ &= \int_2^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-x+1}dx \\ &= \frac{1}{2}\int_2^3 \sqrt{x+1}-\sqrt{x-1}dx \\ &= \frac{1}{2}\int_2^3 \sqrt{x+1}dx - \frac{1}{2}\int_2^3 \sqrt{x-1}dx.\end{aligned}$$

We now perform a pair of translation-substitutions, setting u = x+1 in the first integral and u = x-1 in the second integral:

$$\begin{aligned}&= \frac{1}{2}\int_3^4 \sqrt{u}du - \frac{1}{2}\int_1^2 \sqrt{u}du. \\ &= \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Big|_3^4 - \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Big|_1^2 \\ &= \frac{\sqrt{64}-\sqrt{27}-\sqrt{8}+1}{3}.\end{aligned}$$

$\square$