MA 1B RECITATION 01/08/15

1. Introduction

Hi, I'm Brian, and I'll be your TA this quarter. Here's some basic info

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- My office: 158 Sloan
- My office hours: 4pm Sunday afternoon, or by appointment.
- Section website: http://hwang.caltech.edu/ma1b/

Recitation notes will be posted on the section website within a few days of the recitation itself.

Late policy: You are allowed a maximum of one extension on the homework (for at most a week after the assignment it due). If you need to use it, just send me an email before the due date. Late assignments beyond this will require a note from the Dean or the Health Center—-but don't be afraid to ask for these if you really need it.

1.1. **The Big Picture.** The practical section usually begins by doing tons of matrix computations. The analytic section starts off with the abstract definition of a vector spaces. How are these two objects related?

The primary reason is that matrices are explicit realizations of maps between vector spaces. For instance, a map between two 2-dimensional vector spaces can be represented by a 2-by-2 matrix.

The main objects of study in linear algebra are *linear transformations*, that is, linear maps between vector spaces.

We are still studying the same object, but we are taking two different paths. The practical section takes a "bottom-up" approach, getting comfortable with the formalism of matrices before learning what they are supposed to represent, then learning the real definitions to understand what they were really doing when they were manipulating matrices. Here, in the analytic section, we take a "top-down" approach, where we try and tell you the whole story up-front, then proceeding systematically, building abstraction upon abstraction until we can prove nontrivial results in just a couple of lines. This tends to require a little more mathematical maturity and self-initiative, but it allows us to proceed through more material and to have a better understanding of how to approach a problem from the perspective of a mathematician, which is why this class remains in the Caltech Core.

1.2. **Philosophy of Linear Algebra.** What distinguishes linear algebra from other math courses that you might have taken before is that it comes off as "austere." Especially in the beginning, you get definition after definition in rapid succession, and if you have no idea what it's for or why it is useful, it can seem very dry.

The power of algebra—and linear algebra in particular—is that by carefully constructing the tower of abstractions, you can prove highly nontrivial things in a

Date: January 8, 2015.

very concise way. Therefore, **memorizing the definitions perfectly** is crucial to understanding the material.

Unlike calculus courses, where you could essentially get away with a vague idea of the definitions and facility with the techniques, this will not suffice for a proof-based linear algebra course, like this one. As an analogy, to know the definition of "derivative" of a function f at x perfectly is not only to know it as the vague notion of "rate of change" or "slope of the tangent," but as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. I know this sounds nitpicky, but it's hard to overemphasize this point, and failure of this is almost universally the reason students do not do well in this course.

The best way to memorize the definitions is to **grasp the concepts at an intuitive level**. Aside from practicing these techniques in homework assignments, a good way to practice this is to ask yourself the following two questions whenever you see a definition:

(1) What phenomenon is this definition trying to model?
(2) Why is this *particular* definition the best way to model it?

The abstractions in linear algebra have been codified and solidified over centuries of use and the definitions that we teach you in class are the ones that have been tested over time for maximum power and generality. However, since we are so far removed from the motivating phenomena, it is up to us to fill in the details, to at take a couple of seconds to think about the where this object came from and where we might be going by studying it.

2. What is a vector space supposed to model?

Well, as you might have guessed, they're supposed to model vectors, the familiar vectors you remember from before college (or your physics classes).

What is a vector? It a quantity with a direction and magnitude. The concept of vector space is trying to generalize that of a set of vectors in some Euclidean space, like $\mathbf{R}^2$.

The intuition still holds to some extent for $\mathbf{R}^n$, but it does not work so well for arbitrary vector spaces, as we'll see with some examples that don't look like $\mathbf{R}^n$. With this, I want to emphasize that it's probably best to just view elements of a vector space as objects satisfying the vector space axioms, although you can use the "arrow" intuition as a first approximation.

Example 1. The set of continuous functions $f : \mathbf{R} \to \mathbf{R}$ form a real vector space, with vector addition and scalar multiplication being just the usual addition of functions and multiplication by real constant.

For those of you who are familiar with most the material today and want to think of something challenging, try and answer the following question. What is a basis for this vector space over $\mathbf{R}$? The answer is not important for this class, but it illustrates one reason why we want to work with vector space axioms instead of "intuitively" using the "vectors as arrows."

Example 2. Here's a weird example of a vector space. Consider the positive real numbers $\mathbf{R}_+$. This is a vector space over $\mathbf{R}$. Here 1 is the "zero vector," "scalar multiplication" is numerical exponentiation ($c \cdot v = v^c$), and "addition" is numerical multiplication ($v + w = vw$). What is "$-v$"? It's the real number $1/v$.

It's intuitively "smaller" than the real numbers as a set. What's a basis for this? (Think about properties you know about the real numbers.)

What about some non-examples of vector spaces? We'll see some in the next section, as we talk about subspaces.

3. Subspaces

Recall that a **subspace** is a subset S of a vector space V that is closed under addition and *scalar* multiplication (that is, multiplication by the real numbers for real vector spaces).

Let's start with some examples.

Examples of subspaces in $\mathbf{R}^2$:

- The origin itself.
- Lines through the origin.
- $\mathbf{R}^2$ itself.

Indeed, this is a complete classification of subspaces of $\mathbf{R}^2$. In particular, note that subspaces must always contain the zero element 0, since we can always multiply by 0. Another reason that subspaces must contain a 0 is because subspaces are vector spaces themselves, and so must have some 0.

Remark 1. Note that the empty set $\emptyset$ is not a subspace of a vector space, because it does not contain an additive identity element 0. It also cannot be a vector space, for the same reason.

Non-examples of subspaces in $\mathbf{R}^2$:

- Lines not passing through the origin.
- Two parallel lines (even if one passes through the origin).
- The unit circle or unit-square.
- The closed right half-plane (i.e. points such that $x \geq 0$).
- The union of the (closed) first and third quadrants.

In particular, note that the last example is closed under scalar multiplication but not addition. Thus, we see that subspaces in $\mathbf{R}^2$ have a distinct geometric "shape." We see this phenomenon also holds in $\mathbf{R}^3$.

Examples of subspaces in $\mathbf{R}^3$:

- The origin itself.
- Lines through the origin.
- Planes through the origin.
- $\mathbf{R}^3$ itself.

Much like in the case of $\mathbf{R}^2$, it turns out that this is a complete classification of subspaces in $\mathbf{R}^3$.

Non-examples of subspaces in $\mathbf{R}^3$:

- Any ball, even if it contains the origin or is symmetric about the origin.
- Cone centered at the origin
- Double cone centered at the origin (but note that this is closed under scalar multiplication).

What about $\mathbf{R}^4$? Take the fourth dimension as time. Think about it. Is $\mathbf{R}^3$ a subspace of $\mathbf{R}^4$? What about a plane? (You have to think about time.) What about a plane, for a duration of 5 sections?

How about subspaces of $\mathbf{R}^5$? One way to model the fifth dimension is to use a coloring.

As you can see, once we got beyond things we could easily visualize, it becomes slightly harder to recognize subspaces "on sight" and that is the reason one of the reasons why we use this abstract notion of vector space. With this definition, a vector is merely an object of a vector space, not a tuple of elements. It turns out that they coincide in the case of $\mathbf{R}^n$, since this was a motivating example, but stubbornly sticking to such a notion will get you in trouble later on.

One nice thing about linear algebra is that once you master the definitions, proving something about vector spaces that is true for n dimensions will be as easy as proving it for 2 or 3 dimensions.

4. Polynomials

You've certainly seen these before, but let's review things from the perspective of linear algebra.

Let $F = \mathbf{R}$ (the real numbers) or $F = \mathbf{C}$ (the complex numbers). A **polynomial in** x **over** F is the (formal) sum

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m = \sum_{i=0}^{m} a_i x^i$$

for some nonnegative integer m and $a_i \in F$. We say that it's a "formal" sum, because we are not concerned about issues of convergence, like the problem that we might get infinity if we plug in a specific value for x into $f(x)$.

Remark 1. Note that this is just a strange way to write a sequence $a_0, a_1, a_2, \ldots$ that is "eventually zero", that is, where there exists a number N such that $a_n = 0$ for all $n \geq N$. This is sort of the "raw data" of the polynomial, and the only thing that we are concerned about, when viewed from an algebraic perspective.

The a_i is called the ith **coefficient of f**. We usually insist that $a_m \neq 0$, in which case, we say that the **degree of f** is m, which we denote by $\deg(f) = m$. (Otherwise the degree is simply the number "i" of the highest nonzero coefficient.)

Two polynomials are equivalent if and only if all of their coefficients are the same for every i. (This is obvious, but worth stating because we will deal with other notions of equivalence later on in the course, and these will not necessary be equality.)

Write $F[x]$ for the set of all polynomials (of all degrees). This set admits two binary operations: $+$ and $\cdot$, called **polynomial addition** and **polynomial multiplication**. They are just the addition and multiplication of polynomials that you have been doing since your first algebra class. These operations are both associative and commutative; they also satisfy distributive properties.

Also, note that $F[x]$ admits a **scalar multiplication**, that is, if $c \in F$ and $f \in F[x]$, then

$$c \cdot \sum_i a_i x_i = \sum_i (c a_i) x^i \in F[x].$$

In particular, note that this distributes with addition.

Since $F[x]$ admits (polynomial) addition and scalar multiplication that satisfy certain nice properties, $F[x]$ is a vector space. You can verify that $F[x]$ satisfies the axioms of a vector space.

Remark 2. Note that we didn't even need polynomial multiplication to check that $F[x]$ is a vector space. It comes along as a bonus! But it seems like a waste to lose such a nice operation entirely. Indeed, if you study algebra beyond this course (e.g. if you declare a math or physics major), you will see that $F[x]$ is also a key example of algebraic objects with even more structure, like *rings* and *modules* and *algebras*. But you need not concern yourself about this stuff for this class (unless you want to! in which case you can certainly ask me about them).

Remark 3. Note that $F[x]$ is an infinite-dimensional vector space over F. You don't need to know this information yet, but a possible basis of $F[x]$ over F is given by $1, x, x^2, x^3, \ldots$.

Which of these are subspaces of $F[x]$?
Subsets consists of $f \in F[x]$ such that

- $f(0) = 0$? Yes.
- $f'(0) = 0$? Yes.
- $f''(0) = 0$? Yes.
- $f(0) + f'(0) = 0$? Yes.

Let's prove one of these results, just to review how to go about doing a proof and to establish some standard of what we expect on your homeworks.

Proposition 4. *The set S of polynomials $f \in F[x]$ such that $f(0) + f'(0) = 0$ is a subspace of $F[x]$.*

Proof. If $f(x) = \sum a_i x^i \in F[x]$, then $f(0) = a_0$ and $f'(0) = a_1$, so $f \in S$ if and only if $a_0 + a_1 = 0$. Note that S is nonempty, since the zero polynomial $f(x) = 0$ satisfies $f(0) + f'(0) = 0$. To show that S is a subspace, we need to show that S is closed under addition and scalar multiplication.

Let $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{i=0}^{m} b_i x^i \in S$. To show that S is closed under addition and scalar multiplication, it is enough to show that $cf(x) + dg(x) = \sum e_i x^i \in S$ for any $c, d \in F$. We have

$$e_0 + e_1 = ca_0 + da_0 + ca_1 + da_1 = c(a_0 + a_1) + d(b_0 + b_1) = c \cdot 0 + d \cdot 0 = 0,$$

so S is a subspace, as desired. $\square$