MA 1B RECITATION 01/29/15

1. Announcements

The midterm will be handed out next week.

Midterm review will be in Sloan 151 either next Thursday or Friday at 8pm in Sloan 151. Please answer the poll to indicate which of these times work for you.

Next week will be midterm review, so if you'd like me to cover a specific topic, send me an email, and I'll cook up a problem to cover it next week.

2. Crash Course on Matrix Theory

Most of you have extensive experience manipulating matrices, but now that we've learned the theory, let's learn why the techniques work the way they do. The main question that we'll try to answer today is:

Question 1. *Why can we use Gauss–Jordan elimination ("row reduction") to simplify our matrices? In other words, what is the theory behind the algorithm?*

Let's quickly recall some basics about matrices and their forms.

For row reduction, we are allowed the following elementary row operations: you can

- switch two rows
- multiply a row by a nonzero scalar
- add one row to another

These are represented by the elementary matrices. Note in particular that the elementary matrices are invertible.

A matrix is in **row echelon form** if

- All nonzero rows are above any zero rows.
- The leading coefficient (pivot) of a nonzero row is always to the right of the pivot of the row above it.

These two conditions imply that only zeros must lie under a pivot. Here's a matrix in row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

A matrix is in **reduced** row echelon form if it is row echelon form and

- Every pivot is 1 and is the only nonzero entry in its column.

This is a matrix in reduced row echelon form:

$$\begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Note in particular that we can have nonzero entries in a column, as above, as long as that is not a column with a pivot.

The most important fact, which is surprisingly tricky to prove, is the following result.

Fact 2. The reduced row echelon form of a matrix is unique.

We stop when we reach reduced row-echelon form, because it allows us to easily calculate invariants like the rank; and thus by the rank–nullity theorem, also makes it easy to calculate the nullity.

For instance, what's the rank of the matrix just above?

It's 4. And thus the nullity is 1.

Let's pause here for a conceptual check.

Question 3. *What if we allowed the analogous column operations instead? What if we allowed both row and column operations?*

Using column operations actually gives us an equivalent theory, but then we would secretly be working on the "dual" vector space, which consists of "linear functionals" on a vector space V, that is, linear functions from V into the scalar field (e.g. $\mathbf{R}$ or $\mathbf{C}$). For finite-dimensional vector spaces, the dual vector space is the same dimension as the original vector space, and so they are isomorphic as vector spaces. We'll touch upon this concept later in the course.

Remark 4. The vector space and its dual are not "canonically" isomorphic, in other words, the isomorphism requires taking some choices, so the isomorphism that you choose and that I choose may differ. However, it turns out that a vector space *is* canonically equivalent to its "double dual." Such distinctions lead to some very interesting mathematical properties that have, for example, some physical manifestations.

If we allowed both row and column operations, we wouldn't really have a unique reduced form of the matrix. One could, say, conceive of ways to turn any matrix of the same rank into another matrix of the same rank. Furthermore, we lose the primary application: being able to solve systems of linear equations.

3. Systems of Linear Equations

The matrix manipulation techniques like Gaussian elimination are probably best learned just by working through some examples yourself, rather than having be pontificate about how to approach things, so let's just work through an example with a running commentary.

Example 1. Consider the system of linear equations given by

$$\begin{aligned} 3x + 2y + z &= 1 \\ 5x + 3y + 3z &= 2 \\ x + y - z &= 1. \end{aligned}$$

What are the solutions to this system?

Solution. We will solve this problem by Gauss-Jordan elimination. The associated (augmented) matrix of the system is

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

As a general principle for row reductions, we want to get a 1 as far up and left as possible, and if we already have a nice row with a 1, as in our case, we should get it to the top. So, applying a permutation of the rows, we get

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \end{bmatrix}.$$

We then subtract multiples of the first row from the second and third rows:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 0 & -2 & 8 & -3 \end{bmatrix}.$$

We now multiply the second row by -1 and add twice that row to row three:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the last row in the unaugmented matrix is 0, but the last row of the augmented matrix is nonzero, we see that the system is not consistent, and so has no solution. □

Remark 2. A system of linear equations can only have 0, 1, or ∞ solutions. Using what you know already, can you explain why this is the case? For instance, why can't you have just *two* solutions to a system of linear equations?

4. Invertible Matrices

A matrix A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Note that invertibility only really makes sense for square matrices.

Let's prove an easy proposition about inverses.

Proposition 1. *The inverse of a matrix A, if it exists, is unique.*

Proof. Suppose that A is invertible, so it has an inverse matrix B such that $AB = BA = I$. Suppose that B' is also an inverse of A. We have

$$B = BI = BAB' = IB' = B'.$$

□

Here's a fundamental fact, which is also a part of the Grand Theorem of Linear Algebra.

Fact 2. Any invertible matrix is can be reduced by row operations to the identity matrix.

Every row operation can be realized as left multiplication by an elementary matrix. Row operations can transform A into the identity, that is, we have

$$E = E_N \cdots E_2 E_1$$

where E_i is an elementary matrix, such that

$$EA = I$$

so $E = A^{-1}$. In other words, we have shown the following result.

Proposition 3. *Any invertible matrix can be represented as a product of elementary matrices.*

This is the reason why the algorithm for finding inverses of matrices works. It also implies that the algorithm will fail if the matrix that you're trying to invert is not invertible.

Example 4. Find the inverse of

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0. \end{bmatrix}.$$

The result you should get after running the algorithm should be:

$$\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}.$$

Make sure to check that you do indeed get the inverse by multiplying the two matrices together, because it's very easy to make arithmetic mistakes here.

Here's a question to think about related to this example. Note that the entries of the inverse are pretty large compared to the original matrix.

Question 5. *If all the entries in an invertible matrix A have absolute value $\leq C$ for some constant, can you say anything about the entries in the inverse matrix A^{-1}?*

5. Similarity

We say that two $n \times n$ matrices A and B are similar if

$$B = C^{-1}AC$$

for some invertible $n \times n$ matrix C. By viewing C as the base change matrix, we can see that similar matrices represent the same operator under different bases. Thus, we can split up the set G_n of all invertible $n \times n$ matrices into a disjoint union of sets consisting of similar matrices:

$$G_n = \amalg_\alpha \{A \in M_n \mid A \text{ is similar to } B_\alpha\}$$

where the product runs over a matrix representative B_α of each similarity class (set of matrices that are similar to each other).

Just to test your conceptual understanding:

Question 1. *How big is the collection α?*

It is infinite and is indexed, for instance, by the invertible linear maps between n-dimensional vector spaces with fixed bases on each side.

How can we tell when two matrices are similar? We'll learn more powerful techniques for this soon, but here's one way using only what we know now.

Question 2. *If A is similar to B and A is invertible, must B also be invertible?*

Yes. Because they represent the same linear transformation. What's a way to realize this at the matrix level? Note that since $B = C^{-1}AC$, we have

$$B^{-1} = C^{-1}A^{-1}C.$$

Question 3. *If A is similar to B and A is not invertible, must B also be not invertible?*

Yes. Suppose for contradiction that $B = C^{-1}AC$ is invertible. Then we can write $A = CBC^{-1}$ and since B is invertible, CBC^{-1} would be invertible, thus implying that A is invertible, which is a contradiction.

6. Putting It All Together

Note that if we have an system of n equations in n variables, we can represent it as

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n \times n$ matrix and $\mathbf{x}, \mathbf{b}$ are n-dimensional vectors. Note that if A is invertible, a quick way to solve this system is find the inverse, because then

$$A^{-1}A\mathbf{x} = \mathbf{x} = A^{-1}b.$$

Note that this implies that our system must have a unique solution.

Let's try and put it all together and prove one part of our Grand Theorem. It's fairly amusing because while direction of the proof is easy, the other direction requires throwing pretty much everything we've learned at it to get the result.

Proposition 1. *Let A be an $n \times n$ matrix and $\boldsymbol{x} \in \boldsymbol{R}^n$. The equation $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution (i.e. if A has a pivot in each column in reduced echelon form) if and only if A is invertible.*

Proof. If A is invertible, we see that

$$A^{-1}A\mathbf{x} = \mathbf{x} = \mathbf{0},$$

so this direction is easy.

Now we must prove the converse. Suppose that $A\mathbf{x} = \mathbf{0}$ has a unique solution. Since $\mathbf{x} = \mathbf{0}$ is a solution, this must be the unique one by assumption. By the rank–nullity theorem, this implies that the rank of the matrix is n, that is, it has full rank. Since it has full rank, we know that its reduced row echelon form looks like the identity matrix. Since we can use elementary row operations E_i to reduce A to the identity and elementary matrices are invertible, we have $E = E_N \cdots E_2E_1$ such that $EA = I$, and so $E = A^{-1}$ and thus A must be invertible. □

This isn't the shortest proof or the most direct one, but it does give you an idea of why linear algebra has such powerful results, despite really only one big theorem in the whole class.