MA 1B RECITATION 02/05/13

1. Announcements

The midterm is now available. If you don't have a copy for some reason, pick one up from the Math Office on the second floor Sloan. The midterm is due Sunday night at the same time as the homework.

The midterm review will take place Friday at 8pm in Sloan 151.

2. Interlude on Determinants

You don't need to be completely familiar with determinants for your midterm, and we'll get more in-depth into determinants in the following week, but it may be useful as a nice check for problems you may encounter.

Recall that the determinants are associated to square matrices and are given in the 2×2 and 3×3 cases by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

However, the analogous procedure does not work for 4×4 matrices and higher. We'll have to use other methods to calculate the determinant in this case, which we'll see soon.

An important property about determinants is that they are multiplicative, that is,

$$\det(AB) = \det(A)\det(B).$$

Probably the most important property about determinants it that they give a condition in our grand theorem.

Proposition 1. *For an $n \times n$ matrix A, the following are equivalent:*

- *(a) A is invertible*
- *(b) A has full rank (equivalently, nullspace $\{0\}$)*
- *(c) $\det(A) \neq 0$.*

In low-dimensional cases, checking the determinant is usually the fastest way to check that a map is invertible or has full rank.

It is also useful for doing checks at steps in your work. For instance, all base change matrices are invertible, so they must have nonzero determinant. If you stop and calculate the determinant of such a matrix and you somehow get a zero determinant, you have done something wrong before and need to check your calculations.

Date: February 5, 2015.

3. Midterm review

Example 1. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation be represented in the standard basis $X = \{e_1, e_2\} = \{[1,0]^t, [0,1]^t\} = \{x, y\}$ by

$$m_X(T) = A = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}.$$

What is the matrix representation of T with respect to the basis $Y = \{v_1 = [3,1]^t, v_2 = [1,2]^t\}$.

Solution. The change of basis matrix here is given by

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and so

$$B^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} m_Y(T) &= B^{-1} m_X(T) B \\ &= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 12 & -1 \\ 4 & -2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus, the transformation is determined by

$$\begin{aligned} Av_1 &= 4v_1 + 0v_2 = 4v_1 \quad \text{(first column of } B) \\ Av_2 &= 0v_1 - 1v_2 = -v_2 \quad \text{(second column of } B). \end{aligned}$$

Let's check this:

$$\begin{aligned} Av_1 &= \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 4v_1 \\ Av_2 &= \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -v_2. \end{aligned}$$

□

Example 2. Consider the vector space $P = P_{\leq 2}$ of polynomials of degree ≤ 2. Consider the following two bases of P:

$$\begin{aligned} B &= \{1, x, x^2\} \\ E &= \{1, 1+x, 1+x+x^2\}. \end{aligned}$$

Write the differentiation operator $T : P \to P$ with respect to the bases B and E, respectively.

Solution. We have

$$m_B(T) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Write $E = \{e_1, e_2, e_3\}$, where

$$\begin{aligned} e_1 &= 1 \\ e_2 &= 1 + x \\ e_3 &= 1 + x + x^2 \end{aligned}$$

so the base change matrix from E to B is given by

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A short calculation tells us that the inverse is

$$C^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We then calculate

$$\begin{aligned} m_E(T) &= C^{-1} m_B(T) C \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Alternatively, one could have calculated the matrix representation of T with respect to E directly and obtained the same matrix. If you have time, it is good to try and see whether you get the same matrix using both methods, so that you catch yourself from making any arithmetic mistakes. □

Example 3. Find a basis for the vector space V spanned by $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$ and $w_4 = (1, 1, 1)$.

Solution. Since we have four vectors in $\mathbf{R}^3$ this is obviously a linear dependent set. Can we find a basis within it? (Remember the results we know about bases and spanning sets.) There are a couple of ways to solve this problem, but here's one way to approach it.

To pare down this spanning set, we need to find a relation

$$r_1 w_1 + r_2 w_2 + r_3 w_3 + r_4 w_4 = 0$$

where $r_i \in \mathbf{R}$ are not all zero. Equivalently, we have

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So to find a nontrivial relation, we need to solve this system of equations. We proceed by applying row reduction and try to get things in reduced row echelon form (1's in each column with zeros above):

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we have

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \quad \Leftrightarrow \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

which has a general solution

$$(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0), \quad t \in \mathbf{R}.$$

with a particular solution given by $(2, 1, -1, 0)$. Thus, we have the nontrivial relation

$$2w_1 + w_2 - w_3 = 0.$$

Thus, any of the vectors w_1, w_2, w_3 can be dropped. For instance, $V = \text{span}(w_1, w_2, w_4)$.

But is w_1, w_2, w_4 a basis for V? Let's check whether they these vectors are linearly independent:

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 + 0 + 1 - 0 - 1 - 0 = 1 \neq 0$$

so they are indeed linearly independent. Since $\{w_1, w_2, w_4\}$ is a linearly independent spanning set it is a basis for V and we conclude that $V \cong \mathbf{R}^3$. □

Example 4. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation whose matrix representation (with respect to the standard basis) is given by

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

We want to

(a) Find a basis for $N(T)$.
(b) Is T one-to-one?
(c) Find a basis for the range of T.
(d) Is T onto?

Solution. We want solutions v such that $Av = 0$. By row-reduction, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, a basis for $N(T)$ is $\{[3,-1]^t\}$. Since this is not zero, T is not one-to-one.

Now, let e_1, e_2 be the standard basis for $\mathbf{R}^2$. Then $\text{span}(T(e_1), T(e_2)) = \text{range}(T)$. These are just the columns of A. A basis for the space spanned the columns of A are given by $\{[1,2,3]^t\}$. The dimension of the range of T is 1 and the dimension of $\mathbf{R}^3$ is 3, so T is not onto. □

Example 5. Let V be finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show taht if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists a $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Let U be a subspace of V and $S \in \mathcal{L}(U, W)$. Let $(u_1, \ldots, u_m)$ be a basis of U. Then $(u_1, \ldots, u_m)$ is a linearly independent list of vectors in V, and so can be extended to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ on V. Define $T \in \mathcal{L}(V, W)$ by

$$T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = a_1 S U_1 + \cdots + a_m S u_m.$$

Then $Tu = Su$ for all $u \in U$. □