MA 1B RECITATION 01/17/13

1. Warm-up Question

Consider the vector space of 3-by-3 matrices and consider the matrices in it that are "magic squares," where the rows, columns, and diagonals all add up to the same number, for example

$$\begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}.$$

We can get other easy examples of magic squares by rotating the above matrix, or by considering, things like

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Do magic square form a subspace? It turns out they do! You can easily check that the sum of magic squares is another magic square, and that scaling a magic square by some constant gives you another magic square.

The dimension of 1-by-1 matrices is obviously 1, and the dimension of 2-by-2 matrices is 1 as well, with a basis given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Now, what is the dimension of the space of 3×3 magic squares?

Bonus: What is the dimension of the space of $n \times n$ magic squares for $n > 3$?

2. Quick Recap

Let's quickly recall some basic facts and definitions about dimension and vector spaces.

We say a vector w is a **linear combination** of vectors $v_1, \ldots, v_n$ if there exists scalars $a_1, \ldots, a_n$ such that

$$w = a_1 v_1 + \cdots + a_n v_n.$$

The **span** of vectors $v_1, \ldots, v_n$, denoted $\operatorname{span}(v_1, \ldots, v_n)$ is the set of all vectors which are linear combinations of $v_1, \ldots, v_n$, that is,

$$\operatorname{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots a_n v_n \mid a_i \in \mathbf{R}\}.$$

Equivalently, the span is the smallest subspace containing $v_1, \ldots, v_n$.

Given a vector space V, if $\operatorname{span}(v_1, \ldots, v_n) = V$, we say that the vectors $v_1, \ldots, v_n$ **span** V or **generate** V or that $\{v_1, \ldots, v_n\}$ is a **spanning** or **generating set** for V.

We say that the vectors $v_1, \ldots, v_n$ are **linearly independent** if

$$a_1 v_1 + \cdots a_n v_n = 0$$

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implies that $a_1 = \cdots = a_n = 0$. (Otherwise, we say that $v_1, \ldots, v_n$ are linearly *dependent.*)

Proposition 1. *The vectors $v_1, \ldots, v_n$ are linearly independent if and only if every vector $w \in span(v_1, \ldots, v_n)$ can be written **uniquely** as*

$$w = a_1 v_1 + \cdots + a_n v_n.$$

Proof. ($\Leftarrow$) : Suppose we have constants $a_1, \ldots, a_n$ such that $a_1 v_1 + \cdots + a_n v_n = 0$. Since $0 \in \operatorname{span}(v_1, \ldots, v_n)$ it can be written uniquely by our assumption. Since $a_1 = \cdots = a_n = 0$ is one way to write 0, this is thus the only way to write it. Hence, $a_1 v_1 + \cdots + a_n v_n = 0$ implies that $a_1 = \cdots = a_n = 0$ and so $v_1, \ldots, v_n$ are linearly independent.

($\Rightarrow$): Suppose that

$$w = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$$

for constants a_i and b_j. We want to show that $a_i = b_i$ for all i. We have

$$0 = w - w = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n.$$

Since $v_1, \ldots, v_n$ are linearly independent, $a_i - b_i = 0$, that is, $a_i = b_i$. Hence, every vector $w \in \operatorname{span}(v_1, \ldots, v_n)$ is expressed uniquely as a linear combination of $v_1, \ldots, v_n$. $\square$

A linearly independent spanning set of vectors for V is a **basis** for V.

Here are some basic facts about bases.

- Every vector space has a basis.
- Any generating set for V contains a basis.
- Any linearly independent set of vectors can be extended to a basis.
- Any two bases of V contain the same number of elements.

3. Dimension

Since any two bases of V contain the same number of elements, we say that a vector space V is of **dimension** n if its basis consists of n elements.

Here are a couple of obvious consequences of the definition.

- The dimension of $\mathbf{R}^n$ is n.
- The dimension of a vector space is a nonnegative integer, and there exists a vector space of dimension n for every nonnegative integer n.
- The dimension of a *proper* subspace of a finite-dimensioal vector space is less than the dimension of the whole space.
- If A is a set of m vectors in V and $m < \dim(V)$, then A does not contain a basis.
- If A is a set of m vectors in V, and $m > \dim(V)$, then A is linearly dependent.

Dimension in linear algebra is a pretty "coarse" invariant, but it is pretty much the only invariant of vector spaces that we care about! This is because *any* real (or complex) vector space of dimension n is isomorphic (as a vector space) to $\mathbf{R}^n$, that is, they are essentially the same in the eyes of linear algebra. This is a no-brainer for something like $\mathbf{R}^n$, but there are also some more interesting cases of equivalent vector spaces.

Here are a couple of interesting mathematical objects that are "the same" as vector spaces.

Example 1. Consider the set $\mathbf{R}^4$ and $M_2(\mathbf{R})$ the vector space of 2-by-2 real matrices. They are both 4-dimensional over $\mathbf{R}$

Example 2. Consider the vector spaces of $\mathbf{R}^2$ and $\mathbf{C}$, considered as a real vector space (that is, forget that $\mathbf{C}$ has a magical multiplication). Then both $\mathbf{R}^2$ and $\mathbf{C}^2$ have dimension 2 over $\mathbf{R}$ and so are isomorphic as vector spaces.

Example 3. If V is a real vector space of dimension n, then $\mathcal{L}(V, \mathbf{R})$, the set of linear maps from V to $\mathbf{R}$ is also of dimension n, so the vector space and the real-valued linear maps are "the same" as vector spaces.

We can also do some interesting "operations" with vector spaces. Feel free to use the following result.

Theorem 4. *(Intersection/Sum Dimension Theorem) Let V be a vector space and U, W subspaces of V. Then*

(a) $U + W = \{u + w : u \in U, w \in W\} = span(U, W)$ *is a subspace of* V*, and*
(b) $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

Proof. It is easy to see that $U + W = \text{span}(U, W)$ and so is closed under addition and scalar multiplication and so is a subspace of V.

Let $X = \{x_1, \ldots, x_k\}$ be a basis for $U \cap W$. Then X is independent, so X extends to a basis $Y = X \cup \{y_{k+1}, \ldots, y_n\}$ of U and a basis $Z = X \cup \{z_{k+1}, \ldots, z_m\}$ for W. We claim that $B = Y \cup Z$ is a basis for $U + W$. If so then

$$\dim(U+W) = |B| = k+(n-k)+(m-k) = n+m-k = \dim(U)+\dim(W)-\dim(U\cap W)$$

as desired.

Let $u + w \in U + W$. Since Y and Z are bases for U and W respectively, we can write

$$u = \sum_i a_i x_i + \sum_r b_r y_r \quad \text{and} \quad w = \sum_i d_i x_i + \sum_s c_s z_s$$

and so

$$u + w = \sum_i (a_i + d_i) x_i + \sum_r b_r y_r + \sum_s c_s z_s \in \text{span}(B),$$

and so B is a spanning set for $U + W$. It remains to show that B is independent.

Suppose $0 = \sum_i a_i x_i + \sum_r b_r y_r + \sum_s c_s z_s$ and let $x = \sum_i a_i x_i$, $y = \sum_r b_r y_r$, and $z = \sum_s c_s z_s$. Then $z \in W$ and $z = -(x + y) \in U$ as $x, y \in U$, so $z \in U \cap W$ and hence $z = \sum_j d_j x_j$ as X is a basis for $U \cap W$. Thus

$$\sum_j d_j x_j = z = \sum_s c_s z_s,$$

so as Z is a basis for W. However, $x_1, \ldots, x_k, z_{k+1}, \ldots, z_m$ are linearly independent, so by subtracting one expression of z from the other, we see that $d_j = 0 = c_s$ for all j, s. Hence $z = 0 = -(x + y)$, so by the independence of Y, $a_i = b_r = 0$ for all i, r, and thus B is independent. $\square$

Remark 5. Note that this fails for THREE subspaces, that is, it is NOT the case that

$$\dim(U+V+W) \neq \dim U+\dim V+\dim W-\dim(U\cap V)-\dim(U\cap W)-\dim(V\cap W)+\dim(U\cap V\cap W)$$

This works out set-theoretically, but it doesn't work for vector spaces! Consider, say, three distinct lines in $\mathbf{R}^2$. All intersections have zero dimensions, the left-hand-side is 2, but the right-hand side is 3. The problem is that $(U+V)\cap W \neq U\cap W+V\cap W$.

Remark 6. We now have a notion of "addition" for subspaces. Is there a notion of subtraction? "Multiplication"? "Division"? This may seem like a fairly silly and idle question, but the algebraic framework for much of modern mathematics and theoretical physics actually builds upon such questions. For instance, one way to view the mathematical discipline of "K-theory" is as the study of things you can get once you allow for a certain "addition" and "multiplication" of vector spaces with some additional geometric or topological structure, giving you a sort of "arithmetic" of "geometric" vector spaces. It has been used to attack (and resolve!) some of the deepest questions in math and physics. One particularly spectacular application of the above kind of thinking is in the use of "fiber bundles" in physics in the latter half in the 20th century.

4. Answer to the magic squares question

This is a heuristic solution and is missing many details, but you can turn turn this reasoning into a rigorous proof.

In general a magic square can be considered as an n^2-dimensional vector that satisfies $n + n + 2 - 1$ independent constraints: the rows, columns, and diagonals have the same sum, but the sum of the row equations equals the sum of the column equations, so one of those is redundant.

If $n \geq 3$, the dimension is $n^2 - 2n$, unless $n = 4$, in which case, the size of the basis is $n^2 - 2n - 1$.

For instance, every 3-by-3 magic square can be written as

$$\begin{bmatrix} a+b & a+c-b & a-c \\ a-b-c & a & a+b+c \\ a+c & a+b-c & a-b \end{bmatrix}.$$