MA 1B RECITATION 01/31/13

1. Announcements

Midterm exam will be handed out on Wednesday, February 6th and will be due Monday, February 11.

Next recitation, I will go do a midterm review. Send me an email if you want any particular topic covered.

2. Linear Transformations in Matrix Form

Linear transformations can be represented and analyzed using matrices. Indeed, this was why matrices were developed in the first place!

Let's get a little practice doing this.

Example 1. Let $X = \{x_1, x_2\}$ be the standard basis for a 2-dimensional vector space V and $T \in \mathcal{L}(V)$ with

$$T(x_1) = x_1 + 2x_2$$
$$T(x_2) = 2x_1 - x_2.$$

For $f \in \mathcal{L}(V)$, let $m(f) = m_X(f)$ be the matrix of f with respect to X.

We first calculate $m(T)$ and $m(T^2)$:

$$m(T) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

and since $m(T^2) = m(T)^2$ (Self-check: Why?), we have

$$m(T^2) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

Now, consider a different basis $Y = \{y_1, y_2\}$, where

$$y_1 = x_1 + x_2$$
$$y_2 = x_1 + 2x_2.$$

Q: What is $m_Y(T)$ and $m_Y(T^2)$?

By Theorem 4.6 from the lectures,

$$m_Y(T) = B^{-1} m(T) B$$

where $B = m(g)$ for g the unique (Why?) linear map such that $g(x_i) = y_i$. This B is often called the *base change matrix*, for obvious reasons. Since $g(x_1) = y_1 = x_1 + x_2$ and $g(x_2) = x_1 + 2x_2$, we have

$$B = m(g) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

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What about B^{-1}? Usually it's a quite involved process to calculate B^{-1}, but luckily we're in the simple 2×2 case, which we covered in the first homework assignment, so

$$B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} m_Y(T) = B^{-1}m(T)B &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 10 \\ -2 & -5 \end{bmatrix} \end{aligned}$$

and

$$m_Y(T)^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Lastly, let's check that the calculation is correct, on say, $y_1 = x_1 + x_2$: with respect to Y we have

$$\begin{bmatrix} 5 & 10 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and with respect to X we have

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and indeed

$$T(y_1) = 5y_1 - 2y_2 = 5(x_1 + x_2) - 2(x_1 + 2x_2) = 3x_1 + x_2.$$

Remark 2. One way to remember how to represent the base change matrix is to remember the columns correspond to things in this form, so if $X = \{x_1, \ldots, x_n\}$ is your basis,

$$m(T) = m_X(T) = [T(x_1), T(x_2), \ldots, T(x_n)].$$

where the $T(x_i)$ are column vectors. To remember this, think of T acting on the "column vector" given by $[x_1, \ldots, x_n]^T$. But remember this is only a mnemonic!

Also, the matrix representative necessarily depends on the basis, so always state explicitly which basis you are using to represent the transformation, even if it's just a short remark saying "under the standard basis."

2.1. **Preview: Some bases are better than others.** We had something interesting happening in the last example. For instance, we saw that the matrix representations of T^2 with respect to both bases X and Y were the same. It turns out this is not just a coincidence and there is actually something deeper going on.

This raises the following question. Suppose we have a linear transformation T. Is there any way to choose the "best" basis?

What do we mean by best? Just the basis that makes the problem easy to analyze.

For instance, suppose we have a transformation T represented by the following matrix under the standard basis as

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}.$$

Looks pretty un-uniform and general, and I can't analyze its properties too easily. For instance, if I was captured by an evil mathematician that told me to calculate, say, A^{76} by hand, or die, my life would probably start flashing before my eyes, since I'd inevitably make some arithmetic mistake.

But it turns out that this isn't so tough. Consider the basis $Y = \{y_1, y_2, y_3\}$ given by

$$\begin{aligned} y_1 &= -x_1 + -x_2 + 2x_3 \\ y_2 &= x_3 \\ y_3 &= -x_1 + 2x_3. \end{aligned}$$

Then

$$B = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}, \text{ and so } B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

However, we then have

$$B^{-1}AB = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So calculating A^{76} is actually quite easy! Who says linear algebra has no practical application? It can save your life!

But you might say, hey, I guess you were quite lucky to have an math angel on your shoulder tell you this magical basis. But what if I told you that you, too, can do this, without the interference of ethereal beings?

Finding such a basis is something that we will learn about in the second part of the course, but it's worth thinking a bit by yourself how you might go about finding such a basis. It's actually related to the idea of invariants that I mentioned in the last recitation.

What are the "intrinsic" properties of the transformation? That is, which properties of the transformation are the same regardless of the matrix representation we choose for it? It turns out that this will be the secret to finding the "best" basis.

3. Systems of Linear Equations

The matrix manipulation techniques like Gaussian elimination are probably best learned just by working through some examples yourself, rather than having be pontificate about how to approach things, so let's just work through an example with a running commentary.

Example 1. Consider the system of linear equations given by

$$\begin{aligned} 3x + 2y + z &= 1 \\ 5x + 3y + 3z &= 2 \\ x + y - z &= 1. \end{aligned}$$

What are the solutions to this system?

Solution. We will solve this problem by Gauss-Jordan elimination. The associated (augmented) matrix of the system is

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

As a general principle for row reductions, we want to get a 1 as far up and left as possible, and if we already have a nice row with a 1, as in our case, we should get it to the top. So, applying a permutation of the rows, we get

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \end{bmatrix}.$$

We then subtract multiples of the first row from the second and third rows:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 0 & -2 & 8 & -3 \end{bmatrix}.$$

We now multiply the second row by -1 and add twice that row to row three:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the last row in the unaugmented matrix is 0, but the last row of the augmented matrix is nonzero, we see that the system is not consistent, and so has no solution. □

Remark 2. A system of linear equations can only have 0, 1, or ∞ solutions. Using what you know already, can you explain why this is the case? For instance, why can't you have just *two* solutions to a system of linear equations?

Remark 3. This is one of the earliest instances of the phenomenon of natural "trichotomies" in mathematics. Another naturally occurring one is that of real numbers. Either it is negative, positive, or zero. Another classical one is that there are only three associative algebras over the reals (that is, real vector spaces with a nontrivial nonscalar multiplication operation): the reals $\mathbf{R}$, the complex numbers $\mathbf{C}$, and the (Hamilton) quaternions $\mathbf{H}$. For some slightly less mathematical examples, there the fact that every well-defined yes-or-no question has three possible solutions: "No", "Yes", or "Maybe"/"It depends"/"Don't know". (Indeed, such an idea allows us to formulate quantum computing.) There's also the famous epistemological trichotomy made famous by Donald Rumsfeld: There are knowns, there are known unknowns, and there are unknown unknowns.

You might think, big deal, I can find three things anywhere. But it turns out that you can often characterize such phenomena, and allows you to contrast the examples with, say, the division of integers into evens and odds. Aside from intellectual curiosity, it can be helpful to distinguish things with very little information: for instance, are you operating in a computing world based on the standard bit (on or off) or of a quantum bit or qubit (on or off or "on and off")? Such topics are explored in computer science and mathematical logic and in a certain subfield of it called *model theory*, in particular.