MA 1B RECITATION 02/07/13

1. Announcements

Midterm exam will be handed out on Wednesday, February 6th and will be due Monday, February 11.

The midterm review will be held Sunday, February 10th from 3-4pm in 151 Sloan.

2. Midterm review

Example 1. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation be represented in the standard basis $X = \{e_1, e_2\} = \{[1,0]^t, [0,1]^t\} = \{x, y\}$ by

$$m_X(T) = A = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}.$$

What is the matrix representation of T with respect to the basis $Y = \{v_1 = [3,1]^t, v_2 = [1,2]^t\}$.

Solution. The change of basis matrix here is given by

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and so

$$B^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} m_Y(T) &= B^{-1} m_X(T) B \\ &= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 12 & -1 \\ 4 & -2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus, the transformation is determined by

$$\begin{aligned} Av_1 &= 4v_1 + 0v_2 = 4v_1 \quad \text{(first column of } B) \\ Av_2 &= 0v_1 - 1v_2 = -v_2 \quad \text{(second column of } B). \end{aligned}$$

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Let's check this:

$$Av_1 = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 4v_1$$
$$Av_2 = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -v_2.$$

□

Example 2. Find a basis for the vector space V spanned by $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$ and $w_4 = (1, 1, 1)$.

Solution. Since we have four vectors in $\mathbf{R}^3$ this is obviously a linear dependent set. Can we find a basis within it? (Remember the results we know about bases and spanning sets.) There are a couple of ways to solve this problem, but here's one way to approach it.

To pare down this spanning set, we need to find a relation

$$r_1w_1 + r_2w_2 + r_3w_3 + r_4w_4 = 0$$

where $r_i \in \mathbf{R}$ are not all zero. Equivalently, we have

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So to find a nontrivial relation, we need to solve this system of equations. We proceed by applying row reduction and try to get things in reduced row echelon form (1's in each column with zeros above):

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we have

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \quad \Leftrightarrow \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

which has a general solution

$$(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0), \quad t \in \mathbf{R}.$$

with a particular solution given by $(2, 1, -1, 0)$. Thus, we have the nontrivial relation

$$2w_1 + w_2 - w_3 = 0.$$

Thus, any of the vectors w_1, w_2, w_3 can be dropped. For instance, $V = \operatorname{span}(w_1, w_2, w_4)$.

But is w_1, w_2, w_4 a basis for V? Let's check whether they these vectors are linearly independent:

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 + 0 + 1 - 0 - 1 - 0 = 1 \neq 0$$

so they are indeed linearly independent. Since $\{w_1, w_2, w_4\}$ is a linearly independent spanning set it is a basis for V and we conclude that $V \cong \mathbf{R}^3$. □

Example 3. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation whose matrix representation (with respect to the standard basis) is given by

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

We want to

(a) Find a basis for $N(T)$.
(b) Is T one-to-one?
(c) Find a basis for the range of T.
(d) Is T onto?

Solution. We want solutions v such that $Av = 0$. By row-reduction, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, a basis for $N(T)$ is $\{[3, -1]^t\}$. Since this is not zero, T is not one-to-one.

Now, let e_1, e_2 be the standard basis for $\mathbf{R}^2$. Then $\operatorname{span}(T(e_1), T(e_2)) = \operatorname{range}(T)$. These are just the columns of A. A basis for the space spanned the columns of A are given by $\{[1, 2, 3]^t\}$. The dimension of the range of T is 1 and the dimension of $\mathbf{R}^3$ is 3, so T is not onto. □

Example 4. Let V be finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show taht if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists a $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Let U be a subspace of V and $S \in \mathcal{L}(U, W)$. Let $(u_1, \ldots, u_m)$ be a basis of U. Then $(u_1, \ldots, u_m)$ is a linearly independent list of vectors in V, and so can be extended to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ on V. Define $T \in \mathcal{L}(V, W)$ by

$$T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) = a_1SU_1 + \cdots + a_mSu_m.$$

Then $Tu = Su$ for all $u \in U$. □

Remark 5. Defining $T : V \to W$ by

$$Tv = \begin{cases} Sv, & \text{if } v \in U \\ 0, & \text{if } v \notin U \end{cases}$$

does not work, since this map is not (necessarily) linear.

Example 6. Find a basis for the solution set of the system

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0. \end{aligned}$$

Solution. We have

$$\begin{bmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the only condition imposed by the solution set is that $x_1 = 4x_2 - 3x_3 + x_4$. The solution set is

$$\left\{ \begin{bmatrix} 4x_2 - 3x_4 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_2, x_3, x_4 \in \mathbf{R} \right\} = \left\{ x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid x_2, x_3, x_4 \in \mathbf{R} \right\}.$$

Thus, an obvious candidate for the basis is

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have shown that is spans the space, and you can check that is also linearly independent. □

Exercise 7. Let U and W be five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

Proof. From the addition/sum dimension formula, we have

$$\begin{aligned} 9 &\geq \dim(U + W) \\ &= \dim U + \dim W - \dim(U \cap W) \\ &= 10 - \dim(U \cap W) \end{aligned}$$

and so $\dim(U \cap W) \geq 1$. In particular, $U \cap W \neq \{0\}$. □

Exercise 8. Suppose U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $U \cap W = \{0\}$.

Proof. We have

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Since $\dim(U+W) = 8$, $\dim U = 3$, and $\dim W = 5$, this implies that $\dim(U \cap W) = 0$ and so $U \cap W = \{0\}$. □