MA 1B RECITATION 02/21/13

1. Eigenvectors and Eigenvalues of a Matrix

Recall that given a matrix A, an **eigenvalue** of A is a constant λ such that

$$Av = \lambda v$$

for some nonzero vector v, called a λ-**eigenvector** of A.

To me, the eigenvalues represent the "essence" of a linear transformation. Almost every property you could want to know about a linear transformation can be determined once you know its eigenvalues. For instance, you can easily determine a transformation's determinant, trace, and whether it's invertible just by looking at its eigenvalues.

The most important fact about eigenvalues are the following formulas for the trace and determinant of an $n \times n$ matrix A:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i, \quad \det(A) = \prod_{i=1}^{n} \lambda_i$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (not necessarily distinct). For example, from this formula, we see that A is invertible if and only if all its eigenvalues are nonzero.

Nevertheless, life isn't perfect, and eigenvalues don't tell you everything. For instance, the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues but are not similar and represent two different transformations.

It is generally long but straightforward to determine the eigenvalues for linear transformations between finite-dimensional spaces. The infinite-dimensional case is a another story and has many subtleties. The study of such phenomena led the development of a discipline of mathematics called *functional analysis*, which allowed for the development of the theory of quantum mechanics in the early 20th century.

Example 1. Consider the matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix}.$$

Find its eigenvalues and eigenvectors.

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Solution. To find the eigenvalues of A, we need to find the roots of the characteristic equation for A:

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -3 & 3-\lambda \end{bmatrix} \\ = \lambda^2(3-\lambda) + 1 + 0 - 0 - 3\lambda - 0 \\ = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 \\ = -(\lambda - 1)^3. \end{aligned}$$

Therefore, $\lambda = 1$ is the only eigenvalue of A and λ has multiplicity 3.

After you've done an eigenvalue calculation, it's always good to do a sanity check of some sort. One good way is to check that the trace of your matrix is the sum of your eigenvalues. Indeed, we have

$$\mathrm{Tr}(A) = 3 = 1 + 1 + 1$$

so we're OK.

To find the eigenvectors of λ, we need to find solutions to $A - \lambda I = 0$. We have

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the solution space $(A - \lambda I)v = 0$ is spanned by

$$v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and so is one-dimensional. In other words, the λ-eigenspace is spanned by the single vector $[1, -1, -1]^t$. □

Q: Is A similar to a diagonal matrix?

A: No. We can show this as follows. Suppose that A is similar to a diagonal matrix D. Since similar matrices have the same eigenvalues and multiplicities, D must also have just one eigenvalue, $\lambda = 1$, also of multiplicity 3. However, the only such matrix is $D = I$, and there is no matrix similar to I except itself, so we have a contradiction. Hence, A is not similar to a diagonal matrix.

2. Abstract Eigenvalues and Eigenvectors

It is certainly important to be able to find eigenvalues and eigenvectors of matrices, but like any concept in mathematics, it wouldn't be accepted as a fundamental object of study if it didn't also give you a convenient way to prove some nontrivial properties about what we're really interested in, which in this case is linear transformations.

Example 1. Suppose that $S, T \in \mathcal{L}(V)$. Show that ST and TS have the same eigenvalues.

Proof. Let λ be an eigenvalue of ST, so $STv = \lambda v$ for some $v \neq 0$. We want to show that λ is also an eigenvalue of TS. We have

$$TSTv = T(STv) = T(\lambda v) = \lambda Tv.$$

If $Tv \neq 0$, then we see that λ is an eigenvalue of TS, and we are done.

If $Tv = 0$, then we must have $\lambda = 0$, which implies that T is not invertible. Therefore, TS is not invertible and so also has $\lambda = 0$ as an eigenvalue. Since λ was an arbitary eigenvalue, we have shown that every eigenvalue of ST is an eigenvalue of TS.

To complete the proof, we need to prove if that λ is eigenvalue of TS, then it is an eigenvalue of ST. However, we obtain this by simply exchanging the roles of S and T in the above argument. $\square$

Example 2. Let $T \in \mathcal{L}(V)$ be an invertible matrix and $\lambda \in \mathbf{R}\backslash\{0\}$. Show that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Proof. Suppose that λ is an eigenvalue of T, so there exists a nonzero vector $v \in V$ such that

$$Tv = \lambda v.$$

Applying T^{-1} on both sides, we get

$$v = \lambda T^{-1} v.$$

In other words,

$$T^{-1} v = \frac{1}{\lambda} v$$

so $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

To prove the converse statement, replace T by T^{-1} and λ by $\frac{1}{\lambda}$ and then argue as above. $\square$

Here are a couple of interesting examples where we can find the eigenvectors and eigenvalues without finding the roots of the characteristic equation. This is very useful because for these examples, you probably don't want to manually try and compute the characteristic equation; indeed, in the case of a transformation between infinite-dimensional vector spaces, the characteristic "polynomial" is not a polynomial at all!

Example 3. Let V be an n-dimensional vector space and let $T \in \mathcal{L}(V)$ be given by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n);$$

in other words, T is a linear transformation whose matrix representation (with respect to the standard basis) consists of all 1's. What are the eigenvalues and eigenvectors of T?

Solution. Suppose that λ is an eigenvalue of T. The equation $Tx = \lambda x$ becomes the system of equations

$$\begin{gathered} x_1 + \cdots + x_n = \lambda x_1 \\ \vdots \\ x_1 + \cdots + x_n = \lambda x_n. \end{gathered}$$

Therefore,

$$\lambda x_1 = \cdots = \lambda x_n.$$

Therefore, either $\lambda = 0$ or $x_1 = \cdots = x_n$.

First consider the case that $\lambda = 0$. Then all the equations above reduce to the single equation

$$x_1 + x_2 + \cdots + x_n = 0.$$

Thus, 0 is an eigenvalue of T and the corresponding set of eigenvectors is

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 + \cdots + x_n = 0\}.$$

Now consider the case that $x_1 = \cdots = x_n = t$ for some t. The equations above reduce to the single equation

$$nt = \lambda t$$

If $t = 0$, then we have the previous case, so assume $t \neq 0$. In this case, $\lambda = n$. Hence, n is an eigenvalues of T and the corresponding set of eigenvectors is

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 = \cdots = x_n\}.$$

Since $Tx = \lambda x$ implies that $\lambda = 0$ or $x_1 = \cdots = x_n$, we conclude that T has no eigenvalues other than 0 or n. □

Example 4. Let V be a (countably) infinite-dimensional vector space. Consider the backward shift operator $T \in \mathcal{L}(V)$:

$$T(v_1, v_2, v_3, \ldots) = (v_2, v_3, \ldots).$$

What are the eigenvalues and eigenvectors of T?

Solution. Suppose that $\lambda \in \mathbf{R}$ is an eigenvalue of T. The equation $Tv = \lambda v$ becomes the system of equation

$$\begin{aligned} v_2 &= \lambda v_1 \\ v_3 &= \lambda v_2 \\ v_4 &= \lambda v_3 \\ &\vdots \end{aligned}$$

From this, we can see that we can choose v_1 arbitrarily and then solve for the other coordinate

$$\begin{aligned} v_2 &= \lambda v_1 \\ v_3 &= \lambda v_2 = \lambda^2 v_1 \\ v_4 &= \lambda v_3 = \lambda^3 v_1 \\ &\vdots \end{aligned}$$

Therefore, *every* $\lambda \in \mathbf{R}$ is an eigenvalue of T and the set of corresponding eigenvectors is given by

$$\{(a, \lambda a, \lambda^2 a, \lambda^3 a, \ldots) : a \in \mathbf{R}\}.$$

□

Note that this last example is strange in several ways. Why do you think that is?