MA 1B RECITATION 02/28/13

1. Linear Algebra from 10,000 feet

We have about two weeks before the end of the course, so it's good to begin to consolidate some of the material that we've learned so far.

Fundamental Theorem of Linear Algebra: "row rank = column rank." Much of linear algebra would not be possible, were this not true.

The Grand Theorem of Linear Algebra: Let A be a square $n \times n$ matrix. The following are equivalent:

- $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every n-dimensional vector $\mathbf{b}$
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- A can be row-reduced to the identity matrix
- A has full column rank, that is, the columns of A are linearly independent
- A has full row rank, that is, the rows of A are linearly independent
- A is invertible
- $\det(A) \neq 0$
- A does not have zero as an eigenvalue.

If you remember anything from this class, let it be the result above. Indeed, everything in this class is more or less a corollary of either the Grand Theorem of Linear Algebra above or the Spectral Theorem, which we will cover later today.

2. Orthogonal Matrices

Definition 1. A square matrix O with *real entries* with is said to be **orthogonal** if its columns and rows are orthonormal (i.e. are orthogonal unit vectors).

Equivalent, a matrix O is orthogonal if its transpose is equal to its inverse:

$$O^T = O^{-1}$$

that is,

$$O^T O = OO^T = I.$$

Remark 2. Note that orthogonal matrices have ortho*normal* columns and rows, not just orthogonal!

Special properties of orthogonal matrices:

- O is always invertible with inverse $O^{-1} = O^T$.
- O is always **unitary**, that is $O^{-1} = O^*$.
- O is always **normal**, that is, $O^*O = OO^*$.
- $\det(O) = \pm 1$.
- The product of two orthogonal matrices is orthogonal.

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How to think of an orthogonal matrix? A orthogonal matrix represents a linear transformation on the real numbers that *preserves the dot product* of vectors, that is, for any $\mathbf{v}$ and $\mathbf{w}$ upon which O acts, we have

$$\mathbf{v} \cdot \mathbf{w} = (O\mathbf{u}) \cdot (O\mathbf{v}).$$

What this means is that O represents an *isometry* of $\mathbf{R}^n$, that is, a transformation of $\mathbf{R}^n$ that preserves all distances. Examples include rotations and reflections.

Indeed, all the orthogonal matrices of $\det(O) = 1$ are rotations. (You will see a simple example of this on your homework.)

Example 3. The 2×2 rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Remark 4. An easy mnemonic to remember which sin you need to take the negative of: "Your sins are forgiven in heaven." so the $-\sin\theta$ goes on the top row for rotation in the usual direction!

3. Unitary Matrices

A unitary matrix is the complex-valued generalization of an orthogonal matrix.

Definition 1. A *complex* square matrix U is **unitary** if

$$U^*U = UU^* = I$$

where U^*, as usual, represents the conjugate transpose.

Since a unitary matrix generalizes orthogonal matrices, many of the properties for orthogonal matrices also hold for unitary matrices.

Special properties of unitary matrices:

- Given two complex vectors $\mathbf{v}$ and $\mathbf{w}$, multiplication by U preserves the dot product:
$$\mathbf{v} \cdot \mathbf{w} = (U\mathbf{v}) \cdot (U\mathbf{w}).$$
- U is normal.
- U is diagonalizable, and moreover, U is *unitarily similar* to a diagonal matrix, that is,
$$U = VDV^*$$
where V is unitary and D is diagonal and unitary. Thus, we see that the eigenspaces span the entire space.
- The product of two unitary matrices is unitary.
- $|\det(U)| = 1$. Note that we taking the absolute value. This is also the main reason why U is called "unitary."
- The eigenspaces U are orthogonal.
- Every eigenvalue of U has absolute value 1.

Example 2.

$$\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Indeed, we check

$$AA^* = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I_2.$$

4. Hermitian matrices

Hermitian matrices are the complex-valued generalizations of real symmetric matrices.

Definition 1. A **Hermitian** or **self-adjoint** matrix H is a square matrix with *complex* entries that is equal to its own conjugate-tranpose, that is,

$$H = H^*.$$

Remark 2. In physics and some fields of mathematics (especially ones related to physics), U^* often represents complex conjugation and $U^\dagger$ ("U dagger") is used to denote the conjugate-transpose. However, for this class, we will follow the convention in Apostol, so that U^* represents the conjugate transpose.

The most important property of Hermitian matrices is that like real symmetric matrices, *all the eigenvalues of a Hermitian matrix are real.* However, Hermitian matrices also have a number of other important properties.

Special properties of Hermitian matrices:

- All eigenvalues of H are real.
- The diagonal entries of H are real.
- H is normal.
- H has n linearly independent eigenvectors.
- $\det(H)$ is real.

- The *sum* of two Hermitian matrices is Hermitian.
- The inverse of a Hermitian matrix is Hermitian.
- The product of Hermitian matrices A and B is Hermitian if and only if $AB = BA$, that is, that they commute. In particular, the powers A^k are Hermitian.
- The complex Hermitian matrices do *not* form a vector space over $\mathbf{C}$. For instance, I_n is Hermitian, but iI_n is not, because it has complex (indeed purely imaginary) diagonal.

A good analogy to keep in mind is that taking the adjoint (conjugate transpose) is sort of like complex conjugation on $\mathbf{C}$. A complex number z is real if and only if $z = \overline{z}$, so a Hermitian operator $H = H^*$ is analogous to a real number. This is reflected in some of properties H satisfies, including some of those listed above.

Example 3.

$$\begin{bmatrix} 1 & 2+i & 3 \\ 2-i & 4 & 5i \\ 3 & -5i & 6 \end{bmatrix}$$

Hermitian matrices are especially important in physics, because they are often used to represent things that are observable. This is primarily because of the nice fact that the eigenvalues are always real.

5. Skew-Hermitian matrices

Just like Hermitian matrices, skew-Hermitian matrices generalize real skew-symmetric matrices.

Definition 1. A square matrix with complex entries S is said to be **skew-Hermitian** or **antihermitian** if its conjugate transpose is its negative, that is,

$$S^* = -S.$$

Special properties of skew-Hermitian matrices:

- All entries on the diagonal must be purely imaginary.
- The eigenvalues of S are all purely imaginary.
- S is normal.
- S is diagonalizable and has orthogonal eigenvectors for distinct eigenvalues.
- Both iS and $-iS$ are Hermitian.
- S^k is Hermitian if k is even and skew-Hermitian if k is odd.
- If A and B are skew-Hermitian and $a, b \in \mathbf{R}$, then $aA + bB$ is skew-Hermitian.
- $\exp(S) = \sum_{n=0}^{\infty} \frac{1}{n!} S^n$ is unitary.

A neat fact is that an arbitrary square matrix M can be uniquely written as the sum of a Hermitian matrix H and a skew-Hermitian matrix S:

$$M = H + S$$

where

$$H = \frac{1}{2}(M + M^*), \quad S = \frac{1}{2}(M - M^*).$$

Thus, following the analogy with Hermitian matrices above, the skew-Hermitian matrices act like the purely imaginary part of complex numbers.

Example 2.

$$\begin{bmatrix} i & 2 \\ -2 & i \end{bmatrix}$$

These definitions may seem like arbitrary things to learn, but they are incredibly important and show up everywhere matrices are found, from physics to computer science. For instance, the Standard Model of particle physics, our current best theory for the universe, has a *gauge group* (roughly, a representation of the symmetry of the model) of the form

$$U(1) \times SU(2) \times SU(3)$$

This is a $1 + 3 + 8 = 12$-dimensional vector space where the factors represent the 1 photon, 3 vector bosons, and 8 gluons of the Standard Model. But you now know what these letters mean! The U stands for the unitary matrices, and SU's represent the *special* unitary matrices, that is, the unitary matrices of determinant 1.

6. Spectral Theorem

As a preview of things to come (but not in this course), the following result is one of the crowning achievements of linear algebra. Among other things, it tells us *exactly* when we can diagonalize a matrix.

Theorem 1. *((Complex) Spectral Theorem) Suppose that V is a (finite-dimensional) complex vector space with an inner product (e.g. n-dimensional complex vector space with the dot product) and $T \in \mathcal{L}(V)$. Then V has an orthonormal basis consisting of eigenvectors of T if and only if T is normal.*

Remark 2. Warning! This is a relatively high-level result that is (unfortunately) usually not covered in a first linear algebra class. The problem is that the proof involves defining things like "inner product space" in a rigorous way. Indeed, our textbook does not contain this incredibly useful result.

However, while you *probably can't directly use this result on your homeworks or exams*, it's something that is good to know. For instance, know that if you're, say, trying to diagonalize a matrix and it's NOT normal, then you will always fail!

Here's one example of the kind of results that follow from the theorem. Note that with this, we can prove a fairly significant result in just a couple of lines.

Example 3. A normal operator on a (finite-dimensional) complex inner-product space V is self-adjoint if and only if all its eigenvalues are real. (This result strengthens the analogy, in the case of normal operators, between Hermitian matrices and real numbers.)

Proof. Let $T \in \mathcal{L}(V)$ be normal. If T is self-adjoint, then all its eigenvalues are real. Conversely, suppose that all the eigenvalues of T are real. By the complex spectral theorem, there is an orthonormal basis $(e_1, \ldots, e_n)$ of V consisting of eigenvectors of T. Therefore, there exists real numbers $\lambda_1, \ldots, \lambda_n$ such that $Te_j = \lambda_j e_j$ for $j = 1, \ldots, n$. The matrix of T with respect to the basis $(e_1, \ldots, e_n)$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal. The matrix equal equals its conjugate transpose and so $T = T^*$. Hence, T is self-adjoint. $\square$