MA 1B RECITATION 03/07/13

1. Sundry Questions

Since this is the last recitation, I will just go over some problems that will hopefully get you to think about the material in some new ways. The final will be handed out next Wednesday and will be due the following Monday. Good luck!

Example 1. Show that 1 is an eigenvalue of every square matrix with the property that the sum of the entries in each row equals 1.

Proof. Suppose that A is such an $n \times n$ matrix. Let x be an $n \times 1$-matrix, all of whose entries are 1. Then $Ax = x$, so 1 is an eigenvalue of A. □

Example 2. Suppose that $T \in \mathcal{L}(\mathbf{R}^3)$ is an operator with matrix

$$\begin{bmatrix} 51 & -12 & -21 \\ 60 & -40 & -28 \\ 57 & -68 & 1 \end{bmatrix}.$$

Suppose that I tell you that -48 and 24 are eigenvalues of T. (This is indeed the case.) What is the third eigenvalue of T? (Do so without using a computer, calculating a characteristic polynomial, or even writing anything down.)

Proof. The trace of the matrix is 12. Since the trace is the sum of the eigenvalues and the sum of the given eigenvalues is -24, the third eigenvalue of T must be 36. □

Example 3. Consider a 2×2-matrix of real numbers

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that A has a real eigenvalue if and only if

$$(a-d)^2 + 4bc \geq 0.$$

Proof. A number $\lambda \in \mathbf{R}$ is an eigenvalue of A if and only if there exists number $x, y \in \mathbf{R}$ not both zero, such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

This is equivalent to the system of equations

$$(a-\lambda)x + by = 0$$
$$cx + (d-\lambda)y = 0.$$

This system has a solution other than $x = y = 0$ if and only if

$$(a-\lambda)(d-\lambda) = bc$$

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which is equivalent to the equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

There is a real number λ satisfying the equation above if and only if the discriminant of the polynomial is positive, that is, if

$$(a+d)^2 - 4(ad - bc) \geq 0.$$

Therefore, by expanding the inequality, we see that A has a real eigenvalue if and only if

$$(a-d)^2 + 4bc \geq 0.$$

□

Exercise 4. Let V be a real vector space, $T \in \mathcal{L}(V)$ a self-adjoint operator. Show that $\mathrm{Tr}(T^2) \geq 0$.

Proof. Since T is a self-adjoint operator, its matrix representation A is Hermitian. Therefore, V has a basis $(v_1, \ldots, v_n)$ consisting of eigenvectors of T, so there exists $\lambda_i \in \mathbf{R}$ such that

$$Tv_j = \lambda_j v_j$$

for each j. The matrix of T^2 with respect to this basis is

$$\begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}.$$

Therefore, $\mathrm{Tr}(T^2) = \lambda_1^2 + \cdots + \lambda_n^2 \geq 0$. □

Example 5. Suppose that $T \in \mathcal{L}(V)$ has the same matrix with respect to every basis of V. Prove that T must be a scalar multiple of the identity operator.

Proof. We begin by proving that (v, Tv) is linearly dependent for every $v \in V$. Fix $v \in V$. Suppose for contradiction that (v, Tv) is linearly independent. Then we can extend (v, Tv) to a basis $(v, Tv, u_1, \ldots, u_n)$ of V. The first column of the matrix of T with respect to this basis is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now, $(2v, Tv, u_1, \ldots, u_n)$ is also a basis of V. The first column of the matrix of 2 with respect to this basis is

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, T has different matrices with respect to the two bases we have considered, a contradiction. Therefore, (v, Tv) is linearly dependent for every $v \in V$. Therefore, every vector in V is an eigenvector of T.

To finish our proof, we need the following lemma.

Lemma 6. *If every vector is an eigenvector of T, then T is a scalar multiple of the identity operator.*

For each $v \in V$, there exists a constant a_v such that

$$Tv = a_v v.$$

Since $T0 = 0$. We can choose a_0 to be any number, but for $v \in V \backslash \{0\}$, the value of a_v is uniquely determined by the above equation.

To show that T is a scalar multiple of the identity, we must show that a_v is independent of v for $v \in V \backslash \{0\}$. To do this, let $v, w \in V \backslash \{0\}$. We want to show that $a_v = a_w$. We consider two cases.

First suppose that v and w are linearly dependent. Then there exists a constant b such that $w = bv$ and therefore

$$a_w w = Tw = T(bv) = bTv = b(a_v v) = a_v w.$$

Hence, $a_v = a_w$ as desired.

Now suppose that v and w are linearly independent. Then

$$a_{v+w}(v + w) = T(v + w) = Tv + Tw = a_v v + a_w w$$

and so

$$(a_{v+w} - a_v)v + (a_{v+w} - a_w)w = 0.$$

Since v and w are linearly independent, this implies that $a_{v+w} = a_v$ and $a_{v+w} = a_w$ and so $a_v = a_w$ as desired. □

Example 7. If A is a skew-Hermitian matrix, prove that both $I - A$ and $I + A$ are nonsingular and that $(I - A)(I + A)^{-1}$ is unitary.

Proof. Since A is skew-Hermitian, it is diagonalizable, so there exists an invertible matrix C such that

$$C^{-1}AC = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Since A is skew-Hermitian, λ_i's must be purely imaginary. Write $\lambda_k = b_k i$ where b_k is a real number. Then

$$\begin{aligned} D &:= C^{-1}(I + A)C \\ &= C^{-1}C + C^{-1}AC \\ &= I + \text{diag}(\lambda_1, \ldots, \lambda_n) \\ &= \text{diag}(1 + b_1 i, \ldots, 1 + b_n i). \end{aligned}$$

In particular, $1 + b_j i \neq 0$, so D must be invertible. Since D is is similar to $I + A$, we see that $I + A$ must also be invertible. Since $-A$ is also skew-Hermitian, the same argument shows that $I - A$ is invertible.

Since A and I commute, $I + A$ commutes with $I - A$, and hence $I + A$ also commutes with $(I - A)^{-1}$. Let

$$B = (I - A)(I + A)^{-1}$$

and then

$$B^{-1} = (I + A)(I - A)^{-1} = (I - A)^{-1}(I + A).$$

Recall that $(ST)^* = T^*S^*$. Furthermore, we have $(S+T)^* = S^* + T^*$ and if S is invertible, then $(S^*)^{-1} = (S^{-1})^*$. Therefore, since $A^* = -A$, we have

$$\begin{aligned} B^* &= ((I-A)(I+A)^{-1})^* \\ &= ((I+A)^{-1})^*(I-A)^* \\ &= ((I+A)^*)^{-1}(I-A)^* \\ &= (I+A^*)^{-1}(I-A^*) \\ &= (I-A)^{-1}(I+A) \end{aligned}$$

so $B^{-1} = B^*$ and so B is unitary. □

Example 8. If $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V. Define $S, T \in \mathcal{L}(V)$ by

$$S(a_1e_1 + \cdots + a_ne_n) = a_2e_1 - a_1e_2$$

and

$$T(a_1e_1 + \cdots + a_ne_n) = a_2e_1 + a_1e_2.$$

A simple calculation shows that

$$S^*(a_1e_1 + \cdots + a_ne_n) = -a_2e_1 + a_1e_2.$$

From this, we see that $SS^* = S^*S$.

Another simple calculation shows that T is self-adjoint. Thus, both S and T are normal. However, $S+T$ is given by the formula

$$(S+T)(a_1e_1 + \cdots + a_ne_n) = 2a_2e_1.$$

A simple calculation verifies that

$$(S+T)^*(a_1e_1 + \cdots + a_ne_n) = 2a_1e_2.$$

Another calculation shows that $(S+T)(S+T)^* \neq (S+T)^*(S+T)$. Thus, $S+T$ is not normal. Hence, the set of normal operators on V is not closed under addition and so is not a subspace of $\mathcal{L}(V)$. □