MA 1C RECITATION 04/02/15

1. Introduction

I'm Brian, and I'll be your Ma 1c TA this quarter. Here's some basic info.

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Office Hours: Sundays 9-10pm. There are several TAs for the class, be sure to check out other times if you need some help!

Homework is due **Mondays at 10am**. Please do not be late. You get one free extension of at most a week, if you email me the night before it's due.

Don't worry about writing down everything I say. To make the best use of your time, try to follow the flow of the recitation rather than trying to take perfect notes. I will post lecture notes on the section website. I'll often add some comments or afterthoughts, so you might want to check them out even if you take good notes.

Multivariable calculus is a deep and beautiful field with many aspects, and it's impossible to cover everything you'll need in the future in a single quarter, but we'll try our best to at least cover the essentials so that you can pick up what you need for your personal needs. My philosophy towards recitations is to focus on the practical skills that you'll need to do the homework. All the beautiful theory and context will be left to Prof. Kechris in your regular lectures. Here, I'll focus on the essentials, and emphasizes the quick, dirty skills that you will need to succeed in the class. I'll also focus on the trickier material and try to avoid the more routine things that you can easily look up.

On the surface, multivariable calculus seems like a straightforward generalization of single-variable calculus that we covered in Math 1a, and that is probably the best way to view the material. However, as we will see, there are many subtleties that come up. It's not just a simple matter of adding another integral sign to our calculations or taking partial differentials instead of (total) differentials.

I also personally recommend one of the supplementary texts, Schey's *Div, Grad, Curl, and All That*. It was by reading this textbook and doing the exercises that multivariable calculus really "clicked" for me. One cool thing is that it connects what we learn in class to physics, which is a good preview for what you'll hopefully see next year in the 2nd year physics core and to get an idea of what it's like to be "in the trenches" in developing mathematics inspired by physics. If you have similar inclinations to myself and like geometric or physical explanations for mathematical structures or phenomena, you will also like this book. However, it's far from essential for the course.

Date: April 2, 2015.

2. Limits

You certainly got practice with limits of sequences in one real variable, and the higher dimensional cases are similar, except that you replace the absolute value with a norm. For instance, we have the following definition for limits of sequences.

Definition 1. If $\{\mathbf{x}_i\}$ is a sequence in $\mathbf{R}^n$, then we say that $\lim_{i\to\infty} \mathbf{x}_i = L$ if for any $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $||\mathbf{x}_i - L|| < \epsilon$ for all $i > N$.

Limits and continuity are closely interrelated. One reason why they are important is that if a function f has a limit at a point, then f can sometimes be "made continuous" at that point by setting its value to its limit. (Think of a smooth graph in the xy-plane but with a discontinuous jump at one point.) The formal definition is as follows.

Definition 2. A function f (on $\mathbf{R}^n$) has a limit L at $\mathbf{a}$, which we denote by $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $||\mathbf{x} - \mathbf{a}|| < \delta$, then $||f(\mathbf{x}) - f(\mathbf{a})|| < \epsilon$.

2.1. How to prove something has a limit.

You show that a limit exists by exhibiting an N (for sequences) or a δ (for functions) for any given ϵ.

A good way to show that a limit *does not* exist at some point a, is to find two sequences $\{x_i\}$ and $\{y_i\}$ approaching a such that $\{f(x_i)\}$ and $\{f(y_i)\}$ have different limits. Note that this is not a good way to showing that there is a limit, since there are infinitely many sequences that approach a, and unlike the case of $\mathbf{R}^1$, in $\mathbf{R}^2$ and above, there are infinitely many ways to approach a point, not just 2.

Let's see an example of each.

Example 3. Let $f(x, y) = x^2 + y^2$. Show that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Proof. Let $\epsilon > 0$. Pick $\delta = \sqrt{\epsilon}$. If $||(x, y)|| = \sqrt{x^2 + y^2} < \delta$, we have $f(x, y) = x^2 + y^2 < \epsilon$, as desired. □

Example 4. Show that the function $f(x, y) = \frac{x^2}{x^2+y^2}$ does not have a limit at $(0, 0)$.

Solution. Let $v_i = (1/i, 0)$ and $w_i = (0, 1/i)$ be two sequences in $\mathbf{R}^2$ that tend to $(0, 0)$ as $i \to \infty$. Then $f(v_i) = (1/i^2)/(1/i^2) = 1$ and so $f(v_i) \to 1$ as $i \to \infty$. However, $f(w_i) = 0$ and so $f(w_i) \to 0$ as $i \to \infty$. These limits do not agree, so f does not have a limit at $(0, 0)$. □

3. Continuity

Continuity in one real variable is fairly intuitive, and in multivariable calculus, you can still use your intuition, but things becomes subtle in certain cases. For instance, there are "more ways" that a function can be discontinuous in the higher dimensional case.

Definition 1. We say that a function $f : \mathbf{R}^n \to \mathbf{R}^m$ is **continuous at a point** $\mathbf{a} \in \mathbf{R}^n$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$. We say that a function is **continuous** if it is continuous at all points.

Remark 2. One way to interpret this definition is to simple say that a function has a limit at a point $\mathbf{a}$, and the value of the function at $\mathbf{a}$ is precisely the limit. But note again this is just the same definition as in single-variable calculus.

One easy implication of this is that one way to show that something is *not* continuous at a point is to show that it doesn't have a limit there. For instance, in our second example in the previous section, the function f is discontinuous at $(0,0)$.

For convenience, let's also recall the ϵ-δ definition, which is often useful for proving things.

Definition 3. A function f is **continuous** at $\mathbf{a}$ if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $||\mathbf{x} - \mathbf{a}|| < \delta$, then $||f(\mathbf{x}) - f(\mathbf{a})|| < \epsilon$.

However, as you may recall, it is usually difficult (or at least tedious) to prove that a function is continuous just from the definition. So how should you prove something is continuous?

3.1. **How to prove continuity.** Here's a general algorithm of sorts for proving continuity.

(1) Carefully look at the function (is it a rational function? is it a composition of certain functions? draw it if you can!) and get an idea of where it might be continuous or not.
(2) Use facts from Math 1a (e.g. sin/cos are continuous, polynomial are continuous, etc.) and general properties about continuous functions (e.g. sum of continuous function is continuous, composition of continuous functions is continuous, etc.) to show that the function is continuous at most of the points. After this, you should only have a few points, or certain types of points that you need to show are continuous or not.
(3) Prove whether the function is continuous at these remaining points. To prove continuity at a point, use a δ-ϵ argument. To prove discontinuity at a point, it's usually easiest to show that there exists a sequence $x_n \to x$ such that $\lim_{x_n \to x} f(x_n) \neq f(z)$.

Remark 4. It is particularly easy to show that a function is discontinuous at a point if it is not defined at the point. Then you can simply say that it is not defined at point and not have to prove anything. For instance, $1/x$ is not defined at $x = 0$, and so it is discontinuous at $x = 0$.

Remark 5. Don't make this harder than this needs to be. Freely use facts you know from single-variable calculus. Just say something like "since $1/x$ is a rational function, it is continuous when $x \neq 0$."

Let's go through a couple of examples.

Example 6. Consider the function $f(x,y) = \frac{\sin(x)\cos(y)}{x}$. Where is f continuous?

"Solution". We know that sin, cos, and y (the identity function for y) are defined and continuous for all real numbers. Now, $\frac{1}{x}$ is defined (and continuous) for all $x \neq 0$ since it is a rational function. We then conclude that $f(x,y)$ is defined (and continuous) precisely on the set of points that $x \neq 0$. □

Remark 7. The "proof" above would not be sufficient for full points on a homework assignment. (You probably need a couple more sentences to smooth out the proof.) However, it gives you an idea of the general argument.

Many basic examples will take this form. Sometimes it is enough to just figure out where the function is defined, then apply the above theorems. The following is an example of a function that we can "make continuous" at a point.

Example 8. A priori, the function $xy\sin(1/xy)$ is not continuous at $(0,0)$ because it is not defined at 0. However, if we define its value at $(0,0)$ to be 0, is it continuous? In other words, is the function

$$f(x,y) = \begin{cases} xy\sin(1/xy), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$?

Solution. Yes! Let $\epsilon > 0$. Choose $\delta = \min(1,\epsilon)$. If $||(a,b)|| < \delta$, then $|a| < \delta \leq 1$ and $|b| < \delta \leq 1$ and so $|ab| < \delta < \epsilon$. Since $|\sin(1/ab)| \leq 1$, we have $|ab\sin(1/ab)| < \epsilon$, so the function is continuous at $(0,0)$. □

Many cases are like the ones above. However, there are some interesting subtle cases that require some more work to understand. I'll give some examples and their solutions. If you get stuck on these, feel free to contact me about it.

Example 9. Where is the function

$$f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

continuous?

Solution. The function is continuous everywhere except $(0,0)$.

The interesting thing about this example is that you cannot even "make f continuous" at $(0,0)$ by setting it to a certain value. (Can you show this?) □

Example 10. Consider the function $f(x,y) = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. This function isn't defined at $(0,0)$, and shares the same phenomena as the previous example, in that you cannot define $f(0,0)$ in any way to make it continuous. In this case, it's because if you define a x_n which approaches the origin along a line with angle θ from the x-axis, then $\lim_{n\to\infty} f(x_n) = \theta$!

This shows that we can have strange phenomena in higher dimensions that doesn't occur in the case of one real variable. It also emphasize the fact it is a difficult and subtle problem to establish continuity of a function at a point, since you have to consider sequences that approach your points from infinitely many directions. It was indeed these kind of problems that motivated mathematicians to rigorously define continuity via ϵ-δ arguments and later via topological terminology like open sets.

4. Open Sets

4.1. **Why should I care?** The reason that we care about open sets is that they give a natural way of describing many things in mathematics. For instance, recall that we have used two ways of describing continuity this far in calculus: the $\epsilon - \delta$ definition and the sequence definition. Both of these are equivalent to the open set definition, which I find most intuitive.

It's a little abstract when you first see it, but the language of open sets allows you to rigorously extend your intuition for the $\mathbf{R}$ case to higher dimensional ($\mathbf{R}^n$) cases, as well as to the setting of much of modern mathematics and physics (manifolds[1]).

4.2. **Definitions.** Point-set topology (the subfield of math that includes the study of basic properties of open sets) is all about *balls*. What am I talking about? I mean balls in $\mathbf{R}^n$. More concretely, an (open) ball of radius r about about a point $x \in \mathbf{R}^n$ is the set

$$B_r(x) = \{a \in \mathbf{R}^n \mid |a - x| < r\}.$$

You might also see this written as $B_x(r)$ or $B_x(\epsilon)$ in certain textbooks, but I recommend the above convention. However, make sure that you stick with one consistently and make it clear what letter represents the point and the radius.

- A set $A \subset \mathbf{R}^n$ is **open** if for any point $x \in A$, there exists an (open) ball containing x that lies in A.
- We say a set is **closed** if its complement in $\mathbf{R}^n$ is open. Equivalently, a set A is closed if it contains all its limit points, where a limit point is the limit of a sequence contained in A. (This is worth proving, to test your familiarity with these spaces. Give it a shot! Later on in the course, we'll use this limit point definition of closed sets.)
- The **interior** of a set A is the largest open set contained in A.
- The **boundary** of a set A is the set of points $x \in A$ such that every neighborhood of x contains at least one point of A and one point not in A.

Remark 1. The terminology of open and closed sets isn't the greatest, but it's so entrenched in mathematical language that you can't avoid it. It becomes second nature once you learn the definitions, but make sure that it is clear.

Warning 2. *Sets are NOT doors!* A set can be open, closed, both closed and open, or neither closed nor open.

4.3. **How do you prove that something is open?** An important part of this course is learning how to rigorously argue things. You got some practice in Math 1a and 1b, but it's still worth going over.

You prove that a set is open by the following method: Let $x \in A$. Then argue to produce some r such that $B_r(x) \subset A$. You might have to break it into cases depending on what x is, but once you find such an r for every x, you have shown that every point has a ball around it.

Many of these things you can "see" intuitively, while others might require a little work. Let's quickly run through some examples for subsets of $\mathbf{R}$.

- $(0, 1)$? Open.
- $[9, 10]$? Closed.
- $(1, 2]$? Neither open nor closed.
- $\emptyset$ (the empty set)? Open and closed (because it's vacuously true).
- $\mathbf{R}$? Open and closed.
- $\{1\}$? Closed.
- $(0, 2) \cup (3, 5)$? Open.
- $(0, \infty)$? Open.

[1]We won't touch on this rich and fascinating subject in this class, but as a first approximation, these are spaces that look like open sets of $\mathbf{R}^n$ that are "patched together" in a special way, and it is the synthesis of real analysis and topology that allows one to do analysis on such spaces.

- $[0, \infty)$? Closed.
- $\mathbf{Q}$? Think about this. I'll talk about this shortly.

Remark 3. Note that things get a little funky once we start using infinity. For instance, why don't we use $[0, \infty]$ to denote the closed set? This is because ∞ is not actually in the set of real numbers. If you go on to more advanced mathematics courses, you will study analysis in the "extended real numbers," which do include ∞ and is slightly more subtle than the standard case. However, we will not deal with such things in this course.

Example 4. Sanity check. Is the open ball open? It better be! Let's prove this in the case of the unit ball in the plane $\mathbf{R}^2$, that is,

$$B = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}.$$

Proof. Let $\mathbf{v} = (x, y) \in \mathbf{R}^2$. Let $r = 1 - x^2 - y^2 = 1 - ||\mathbf{v}||$. We claim that $B_{r/2}(\mathbf{v}) \subset B$. Pick a $\mathbf{w} \in B_{r/2}(\mathbf{v})$. Then we have

$$\begin{aligned} ||\mathbf{w}|| &= ||\mathbf{w} - \mathbf{v} + \mathbf{v}|| \\ &\leq ||\mathbf{w} - \mathbf{v}|| + ||\mathbf{v}|| \\ &< r/2 + ||\mathbf{v}|| \\ &= \frac{1}{2}(1 - ||\mathbf{v}||) + ||\mathbf{v}|| \\ &= \frac{1}{2}(1 + ||\mathbf{v}||) \\ &\leq 1, \end{aligned}$$

which proves our claim. □

How did I find this r? The way that I did this is by doing a sort of "lazy evaluation," not assigning a value to r until I am just about the use its value at a crucial point in the argument. In many basic examples, this is a good way to find r if you don't have a good idea of what it should be.

Example 5. What about the rational numbers $\mathbf{Q}$? The rationals are not open. In fact, $\mathbf{Q}$ has empty interior. How can you show this? Let x be any rational number and $r > 0$ be any radius. Then $B_r(x)$ must contain an irrational number and so $B_r(x)$ is not contained in $\mathbf{Q}$. Hence, the rational numbers contain no open sets and so the interior is empty.

What about the irrational numbers $I = \mathbf{R} \backslash \mathbf{Q}$? The same argument works. Thus, we see that the rational numbers are neither open nor closed.

5. Appendix: The δ-ϵ definition of continuity and the open set definition of continuity are equivalent for $\mathbf{R}^n$.

Proposition 1. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a function. In the terminology of open sets, we say that f is **continuous** if the preimage of every open set is open, that is, if $f^{-1}(U)$ is open for every open set $U \subset \mathbf{R}^m$. This definition is equivalent to the ϵ-δ definition of continuous given above.*

Proof. Suppose that $f^{-1}(U)$ is open for every open set $U \subset \mathbf{R}^m$. Let $x \in \mathbf{R}^n$ and $\epsilon > 0$. Then $B_\epsilon(f(x))$ is an open set in $\mathbf{R}^m$ containing $f(x)$. Since f is (open-set) continuous, $V = f^{-1}(B_\epsilon(f(x)))$ is an open set in $\mathbf{R}^n$. Note that V contains x

and so x is an interior point of V. In other words, there exists a $\delta > 0$ such that $B_\delta(x) \subset V$, which implies that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Conversely, suppose that f is (ϵ-δ-continuous) and $U \subset \mathbf{R}^m$ is an open set. We want to show that $f^{-1}(U)$ is open. This is vacuously true if $f^{-1}(U)$ is empty, so assume that $x \in f^{-1}(U)$. Since $f(x) \in U$ and U is open, there exists some $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$. By ($\epsilon$-$\delta$) continuity of f, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$, in other words, x is an interior point of $f^{-1}(U)$. Since this is true for any $x \in f^{-1}(U)$, we conclude that $f^{-1}(U)$ is open, as desired. □

As an exercise, try to show that the sequence definition of continuity is also equivalent to the ϵ-δ definition of continuity and the open set definition of continuity.