MA 1C RECITATION 06/04/15

1. Administrivia

This is last recitation, and the final exam for the class is due Wednesday morning. It only covers stuff from the second half of the course, so you won't see Lagrange multipliers, for example. However, there will be proofs on the final, so if you've forgotten, quickly refresh yourself on how to, say, do $\epsilon - \delta$ proofs.

Remember there are often techniques aside from the standard approach that you can use to solve problems quickly. These are good ways to save time on the exam, so that if you run through the problems and have some extra time, you can try and redo the problem the obvious and straightforward way to be doubly sure of your answer. But remember that this is an exam where you are not allowed to use calculators or computers, so the calculations should be simple. If you find yourself in a long, messy calculation, there is probably an easier way!

The final review for this class will be **Monday 10-11am in Sloan 151**. If you have any particular topics that you would like covered, let me know. Best of luck!

2. Final Exam Review Examples

Example 1. Let V be the top half of the solid unit ball in $\mathbf{R}^3$. Compute

$$\iiint_V x^2 + y^2 + z^2 \, dx \, dy \, dz.$$

Solution. Obvious, we should use spherical coordinates. We note that V is the image of $[0,1] \times [0, 2\pi] \times [0, \pi/2]$ in (ρ, θ, ϕ) coordinate, so

$$\begin{aligned} \iiint_V x^2 + y^2 + z^2 \, dx \, dy \, dz &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} \rho\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho \\ &= \frac{\rho^4}{4}\bigg|_0^1 \times (-\cos(\theta)\bigg|_0^{\pi/2} \times 2\pi \\ &= \frac{\pi}{2}. \end{aligned}$$

□

Example 2. Calculate the surface area of a torus around the circle $x^2 + y^2 = R^2$ with internal radius r. (The internal radius being the radius of the circle that you rotate around the z-axis to get the torus.)

The torus is parametrized by the map $\phi : [0, 2\pi] \times [0, 2\pi] \to \mathbf{R}^3$ with

$$\phi(u, v) = (\cos(u)(R + r\cos(v)), \sin(u)(R + r\cos(v)), r\sin(v)).$$

Date: June 4, 2015.

Solution. This is a surface integral calculation, so let's just do it :

$$\begin{aligned}
\iint_T 1\, dT &= \iint_{[0,2\pi]^2} \left|\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right|\right| \\
&= \iint_{[0,2\pi]^2} ||(-\sin(u)(R + r\cos(v)), \cos(u)(R + r\cos(v)), 0) \times (-r\cos(u)\sin(v), -r\sin(u)\sin(v), r\cos(v))||\, du\, dv \\
&= \iint_{[0,2\pi]^2} ||(r\cos(u)\cos(v)(R + r\cos(v)), r\sin(u)\cos(v)(R + r\cos(v)), -r\sin(v)(R + r\cos(v)))||\, du\, dv \\
&= \iint_{[0,2\pi]^2} \sqrt{(r\cos(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(v)(R + r\cos(v)))^2}\, du\, dv \\
&= \iint_{[0,2\pi]^2} r(R + r\cos(v)) \cdot \sqrt{\cos^2(u)\cos^2(v) + \sin^2(u)\cos^2(v) + \sin^2(v)}\, du\, dv \\
&= \iint_{[0,2\pi]^2} r(R + r\cos(v)) \cdot \sqrt{\cos^2(v) + \sin^2(v)}\, du\, dv \\
&= \iint_{[0,2\pi]^2} R \cdot r + r^2\cos(v)\, du\, dv \\
&= 4\pi^2 Rr.
\end{aligned}$$

□

Example 3. Compute the line integral of $F(x, y, z) = (x^2, z, y)$ around the curve C of intersection of $x^2+y^2+z^2 = 1$ and $x + z = 0$, where C is taken counterclockwise when viewed from above the origin.

Solution. This is the intersection of a sphere and a plane. Let's parameterize it. Note that the plane goes through the origin, and we get it by rotating the xy plane about the y-axis. Rotation about the y-axis is the matrix

$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta). \end{bmatrix}$$

What is θ? We want the image of the x-axis to be inside our plane, so we want $\cos(\theta)+\sin(\theta) = 0$. Thus, $\theta = -\pi/4$ works. Therefore, define

$$A(x, y, z) = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This transformation is just a rigid rotation of $\mathbf{R}^3$, so the circle we want is the image of the circle of radius 1 in the xy plane, i.e. $c(t) = (\cos(t), \sin(t), 0)$. This means our circle is

$$C(t) = A(c(t)) = (\cos(t)/\sqrt{2}, \sin(t), -\cos(t)/\sqrt{2}).$$

Note that this is the wrong orientation, so we have to multiply by -1. Now let's compute the integral:

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^{2\pi} (\cos(t)^2/2, -\cos(t)/\sqrt{2}, \sin(t)) \cdot (-\sin(t)/\sqrt{2}, \cos(t), \sin(t)/\sqrt{2})\, dt \\
&= \int_0^{2\pi} -\cos(t)^2\sin(t)/2^{3/2} - \cos(t)^2/\sqrt{2} + \sin(t)^2/\sqrt{2}\, dt \\
&= 2^{-3/2}(\cos(t)^3/3\Big|_0^{2\pi} - (t/2 + \sin(2t)/4\Big|_0^{2\pi} + (t/2 - \sin(2t)/4\Big|_0^{2\pi} \\
&= 0.
\end{aligned}$$

Alternatively, we could note that $\text{curl}(F) = 0$, and C bounds a disk S in $\mathbf{R}^3$ which is just the plane contained in the sphere. Therefore, by using Stokes's theorem, we have $\int_S \text{curl}(F) \cdot \mathbf{n}\, dS = \int_C F \cdot ds$. The former integral is zero, which implies our result. □

Example 4. Find the area of the region R enclosed by the curve

$$\gamma(t) = (\cos(t), \sin(3t))$$

where $t \in [0, 2\pi]$.

Remark 2.1. Such a curve is called a *Lissajous curve* and originated in the theory of complex harmonic motion. One reason it's interesting is because if you shift how fast you're traversing across the cosine or sine part, say by having $\cos(12t)$ in x-coordinate, you end up with a dramatically different looking curve. They also often appear as "slices" of higher-dimensional objects that correspond to some natural physical motion.

Another way that there objects show up in mathematics is as the projection of a (mathematical) knot[1] from 3-space to the plane. Studying the "shadows" of higher dimensional objects in this way is a common technique in modern mathematics and physics.

Solution. Okay, we're in a plane, asked to take an area of something enclosed by a curve which we don't even have to go through the trouble to parametrize. It's even oriented clockwise! Your first instinct should be to use Green's theorem. We have

$$\mathrm{area}(R) = \iint_R 1\,dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\,d\gamma.$$

Thus, we have

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\,d\gamma &= \int_0^{2\pi} \left(-\frac{\sin(3t)}{2}, \frac{\cos(t)}{2}\right)\cdot(-\sin(t), 3\cos(3t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} \sin(3t)\sin(t) + \cos(3t)\cos(t)\,dt.\end{aligned}$$

OK, so we have a slightly tricky integral. How do we solve this. Trig identities! But which ones? The triple-angle formulas, of course!

$$\begin{aligned}\cos(3t) &= 4\cos^3(t) - 3\cos(t) \\ \sin(3t) &= 3\sin(t) - 4\sin^3(t).\end{aligned}$$

(We don't actually expect you to know these off the top of your head. You won't get something so complicated on an exam.) Therefore, we have

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\,d\gamma &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^4(t) - \sin^4(t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t)(1-\sin^2(t)) - \sin^2(t)(1-\cos^2(t)))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t) + \sin^2(t)\cos^2(t) - \sin^2(t)\cos^2(t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} \cos^2(t) - \sin^2(t)\,dt \\ &= \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt \\ &= 0.\end{aligned}$$

So the answer is zero! Wait, that can't be right. By inspection, we see that the region inside our curve is clearly nonzero. Where did we go wrong? It was in the application of Green's theorem. Recall that the hypothesis of Green's theorem states that it only applies to simple closed curves. The curve γ is closed, but it is *not* simple, because the curve intersects itself. So how do we solve this? We need to break our curve into parts and apply Green's theorem to each component.

Right: If we restrict parameter t of γ to $[-\pi/3, \pi/3]$, we get the right-most part of the curve. Here, γ is oriented counterclockwise, so we can find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} \cos(2t)\,dt = \left.\frac{\sin(2t)}{4}\right|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{4}.$$

[1]same as the knots of string you know and love, but by attaching the two pieces together at the ends

Left: By restricting the parameter t to $[2\pi/3, 4\pi/3]$, we obtain the left-most part of the curve. Here, γ is also oriented counterclockwise, so we can once again find the area by evaluating the integral

$$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} \cos(2t)\,dt = \left.\frac{\sin(2t)}{4}\right|_{2\pi/3}^{4\pi/3} = \frac{\sqrt{3}}{4}.$$

Center: Finally, by restricting the parameter t to $[\pi/3, 2\pi/3] \cup [4\pi/3, 5\pi/3]$, we get the center piece, but here γ is oriented clockwise, so we need to reverse the orientation. We get

$$\frac{1}{2}\int_{2\pi/3}^{\pi/3} \cos(2t)\,dt + \frac{1}{2}\int_{5\pi/3}^{4\pi/3} \cos(2t)\,dt = \frac{\sqrt{3}}{2}.$$

Note that the curve γ here was defined piecewise. This is fine. It still a simple closed curve that is counterclockwise oriented. We just need to break the integral into two parts.

Thus, by summing these areas together, we see that the area of the region R enclosed by the curve γ is $\sqrt{3}$. □

Example 5. Compute the flux of the vector field $F(x, y,) = (x, 1 - y^2, z)$ through the set $x^2 + y^2 = 1$, $|z| \leq 1$, that is, a cylinder of radius 1 and height 2 centered at the origin.

Solution. We can parameterize the cylinder using cylindrical coordinates via

$$f(\theta, z) = (\cos\theta, \sin\theta, z).$$

This has derivative vectors

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= (-\sin(\theta), \cos(\theta), 0)\\ \frac{\partial f}{\partial z} &= (0, 0, 1).\end{aligned}$$

The cross-product is

$$\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial z} = (\cos(\theta), \sin(\theta), 0).$$

Thus, the surface integral is

$$\begin{aligned}\iint_S F \cdot \mathbf{n}\,dS &= \int_{-1}^{1}\int_0^{2\pi} (\cos\theta, 1 - \sin^2\theta, z) \cdot (\cos\theta, \sin\theta, 0)\,d\theta\,dz\\ &= \int_{-1}^{1}\int_0^{2\pi} \cos^2\theta + \sin\theta - \sin^3\theta\,d\theta\,dz\\ &= 2\int_0^{2\pi} \cos^2\theta + \sin\theta - \sin^3\theta\,d\theta\\ &= 2\pi.\end{aligned}$$

□