MA 1C RECITATION 05/07/15

1. Line Integrals

The motivation behind line integrals comes from physics: a line integral measures the total work that a field (e.g. a sci-fi style force field, or a gravitational field) does on you as you move through a path.

Line integrals are actually single-variable integrals, but are only interesting when applied in a multivariable context.

Definition 1.1. For a function $f : \mathbf{R}^m \to \mathbf{R}^n$ and a piecewise-continuous path $\gamma : [a,b] \to \mathbf{R}^m$ that is smooth (i.e. infinitely differentiable) on each piece, we define the **line integral of f along** γ to be

$$\int_\gamma f\, d\gamma := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

the resulting output only makes sense if it is not infinite, so we usually assume that f is bounded on the image of the path C that γ traces out. (This is automatic, for instance, if C is compact.)

Notation 1.2. I'm not a fan of this notation, but it is common, especially in physics, so we need to know it.

We can write everything in coordinates $\mathbf{f} = (f_1, \ldots, f_n)$ and $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$. The line integral becomes

$$\int_a^b \sum_i f_i(\vec{\gamma}(t))\gamma_i'(t)\, dt = \int f_1\, d\gamma_1 + \cdots + f_n\, d\gamma_n.$$

Thus, if we ask you to compute the line integral $\int \frac{x\, dy + y\, dx}{x+y}$ over a curve C, we mean for you to take $\mathbf{f}(x,y) = (\frac{y}{x+y}, \frac{x}{x+y})$ (note the order) and take the line integral of a parametrization γ of the path C.

While the line integral might seem to depend on the path, the following result implies that the integral only depend on the curve traced out by the line itself, and not a particular parametrization.

Theorem 1.3. *Suppose that $\gamma : [a,b] \to \boldsymbol{R}^n$ and $\alpha : [c,d] \to \boldsymbol{R}^n$ are two paths such that (1) α and γ have the same image in $\boldsymbol{R}^n$, (2) $\alpha(a) = \gamma(c)$, and (3) both paths traverse their images with the same orientation (roughly, move in the "same direction"). Then for any function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$, we have*

$$\int_\gamma f \cdot d\gamma = \int_\alpha f \cdot d\alpha,$$

provided that either integral exists.

Date: May 7, 2015.

Remark 1.4. In general, if you are asked to take the integral along a circle or rectangle or something else, unless the problem says otherwise, it's safe to assume that you are supposed to integrate with "positive orientation," that is, in the counterclockwise direction. There are reasons for this convention, but a long screed on it doesn't really belong here.

A short answer is given by the fact that "positive" orientation implicitly assumes "with respect to the standard basis" and so terms in the positive quadrant/octant/$4n$-ant are those that are positive when dotted with a basis element.

Proof Sketch. Let's give an idea of why this is true. The question is really asking is if you have some orientation-preserving diffeomorphism (i.e. differential homeomorphism) $h : [c, d] \to [a, b]$—like $h : [0, 1/2] \to [0, 1]$ with $h(t) = 2t$—do we have $\int_\gamma \mathbf{f} \cdot d\gamma = \int_{\gamma \circ g} \mathbf{f} \cdot d(\gamma \circ h)$? A simple calculation shows us this is the case:

$$\begin{aligned}
\int \mathbf{f} \cdot d(\gamma \circ h) &= \int_c^d \mathbf{f}(\gamma(h(t))) \cdot (\gamma \circ h)'(t)\, dt \\
&= \int_c^d \mathbf{f}(\gamma(h(t))) \cdot \gamma'(h(t))h'(t)\, dt \\
&= \int_{h(c)}^{h(d)} \mathbf{f}(\gamma(u)) \cdot \gamma'(u)\, dt \\
&= \int \mathbf{f} \cdot d\gamma.
\end{aligned}$$

□

From this, it is easy to see that reversing orientation of the path gives the negative of the integral, since then the penultimate integral would go from $h(d)$ to $h(c)$.

Let's work through a simple example.

Example 1. Compute $\int_C -y\, dx + x\, dy$ counterclockwise around the circle C of radius 4 in the plane.

First, we need to parametrize the circle:

$$\gamma(t) = (4\cos(t), 4\sin(t)), \quad t \in [0, 2\pi],$$

so $\gamma'(t) = (-4\sin(t), 4\cos(t))$. The function $\mathbf{f}$ is given as $\mathbf{f}(x, y) = (-y, x)$, so we compute

$$\begin{aligned}
\int_C -y\, dx + x\, dy &= \int_0^{2\pi} \mathbf{f}(\gamma(t)) \cdot \gamma'(t)\, dt \\
&= \int_0^{2\pi} (-4\sin(t), 4\cos(t)) \cdot (-4\sin(t), 4\cos(t))\, dt \\
&= \int_0^{2\pi} 16(\sin^2(t) + \cos^2(t))\, dt \\
&= 32\pi.
\end{aligned}$$

Note that there does not exist a function $g : \mathbf{R}^2 \to \mathbf{R}$ such that $\nabla g = (-y, x)$. If we had this property, it would make many things easier, as we'll see in a few minutes.

2. Line Integrals With Respect to Arc Length

A common thing that we want to do with integrals is to simply integrate a scalar function over some path in n-dimensions. Then we can use our usual intuition of the integral as the "area under" the graph. This is not what a line integral computes, but it is what a line integral *with respect to arclength* computes.

Definition 2.1. Given a scalar field $f : \mathbf{R}^n \to \mathbf{R}$ and a continuous path $\gamma : [a, b] \to \mathbf{R}^n$, we can define the **line integral with respect to arc length** of f along γ as

$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot ||\gamma'(t)||\, dt.$$

This is a slight generalization of the notion of line integral, since if we take the scalar function $f(\gamma(t)) \cdot \gamma'(t)/||c'(t)||$, we recover the first definition.

These integrals tend to be harder, because they have a square root factor from the $||\gamma'(t)||$. If you want to make your life easier, try and pick a function that moves at a constant speed.

Just like with the normal line integral, this only depends on the curve drawn by γ and not any particular parametrization. What's going on here is that the arclength fudge factor $||\gamma'(t)||$ forces the integral to go along the curve at a uniform rate, so that it doesn't matter, if you, say, move along the curve at a given rate or at three times that given rate.

Let's see an example of this. Everything is just computational.

Example 2. Integrate the function $f(x, y, z) = x^2y^2 + y^2z^2 + x^2z^2$ over the helix $\gamma(t) = (\cos(t), \sin(t), t)$ where $t \in [0, 2\pi)$.

Proof. We just need to apply our definition above.

$$\begin{aligned}
\int_\gamma f(x, y, z) \cdot d\gamma &= \int_0^{2\pi} f(\cos(t), \sin(t), t) \cdot ||(\cos(t), \sin(t), t)'||\, dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2\sin^2(t) + t^2\cos^2(t)).||(-\sin(t), \cos(t), 1)||\, dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t) + t^2) \cdot \sqrt{\sin^2(t) + \cos^2(t) + 1^2}\, dt \\
&= \int_0^{2\pi} \left(\frac{\sin^2(2t)}{4} + t^2\right)\sqrt{2}\, dt \\
&= \int_0^{2\pi} \left(\frac{1 - \cos(4t)}{8} + t^2\right)\sqrt{2}\, dt \\
&= \left(\frac{t}{8} - \frac{\sin(4t)}{32} + \frac{t^3}{3}\right)\sqrt{2}\Big|_0^{2\pi} \\
&= \left(\frac{2\pi}{8} - \frac{0}{32} + \frac{(2\pi)^3}{3}\right)\sqrt{2} - 0 \\
&= \frac{2\pi\sqrt{2}}{8} + \frac{8\pi^3\sqrt{2}}{3}.
\end{aligned}$$

□

3. Line Integrals and Gradient Fields

Line integrals over gradient fields are especially easy to compute and occur in many natural contexts, so it worthwhile to spend some time on this special case. The first major result is a version of the fundamental theorem of calculus for line integrals.

Theorem 3.1. *(Apostol Theorem 10.3) If ϕ is a differentiable scalar field with a continuous gradient on an open, connected set, then for any path γ with endpoints $\gamma(0) = a$ and $\gamma(1) = b$, we have*

$$\int_0^1 \nabla\phi \cdot d\gamma = \phi(b) - \phi(a).$$

Proof. The function $\phi(\gamma(t))$ is a map $\mathbf{R} \to \mathbf{R}$ with (single-variable) derivative $\nabla\phi(\gamma(t))\cdot\gamma' 9t)$, so it the result follows from the single-variable fundamental theorem of calculus. □

You may have noticed that many line integrals over closed curves often give the result zero. There's a good reason why this happens (it's a corollary of the Theorem above, for instance), and a more conceptual explanation comes from the following important and powerful result.

Theorem 3.2. *(Apostol Theorem 10.5.) Suppose that $S \subseteq \boldsymbol{R}^n$ is an open and connected[1] set. Then, the following conditions are equivalent (that is, if one of the statements hold, then all of the statements hold and if one of these statements doesn't hold, then none of these statements hold), for any function $f : S \to \boldsymbol{R}^n$:*

(a) There is a scalar field $F : S \to \boldsymbol{R}$ such that $\nabla F = f$.

(b) The line integral of f over any path $\gamma : [a, b] \to S$ only depends on the endpoints of γ, that is,

$$\int_\gamma f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a)).$$

(c) The line integral of f over any closed path $\gamma : [a, b] \to S$ (i.e. any path γ with $\gamma(a) = \gamma(b)$, is identically zero.

Since we don't have a rigorous way to talk about "all of the paths" γ in a space S yet, the way we usually apply this theorem is to (1) notice that a given function is a gradient, and then (2) deduce that an other-difficult integral is trivially given by evaluating f on its endpoints, or is zero (because the curve is closed).

Example 3. Recall our example from before. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}\right),$$

what is the integral of f around the circle C_r of radius r traversed counter-clockwise?

Proof. Noting that f is the gradient of the function $F(x, y) = \log(x^2 + y^2)$ and that C_r is a closed curve in an open connected set ($\mathbf{R}^2$ itself), we apply the theorem to see that

$$\int_{C_r} f \cdot dC = 0.$$

[1] means that given any two points in the set, there is a path connecting the two points that also lies in the set

□

Now, we don't have any surefire methods yet in this course for finding such gradients; mostly it's just recognizing patterns and making intelligent guesses. Failing that, you can always do it the long, straightforward way to get the answer.

However, what's really cool about the theorem is that this works in general. If we saw the above example, we might have just tried to calculate the answer, instead of using the theorem. But we can apply the theorem to curves that we would not want to integrate by hand.

Example 4. Find the line integral of the vector field

$$f(x, y) = (yz, xz, xy)$$

over the curve

$$\gamma(t) = (1, \cos(t), W(t)), \quad t \in [0, 2\pi]$$

where $W(t)$ is a Weierstrass function, defined as

$$W(t) = \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi t)}{2^n}.$$

What's cool is that $W(t)$ is an example of a function that is everywhere continuous but nowhere differentiable.

Proof. If you want to do that directly, good luck!

For those of us that don't want to integrate an infinite sum of cosines, we can simply note that because $\cos(0) = \cos(101^n \cdot 2\pi \cdot 0) = 1$, we have

$$\gamma(0) = (1, \cos(0), W(0)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1)$$

$$\gamma(2\pi) = (1, \cos(2\pi), W(2\pi)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1),$$

and so this curve is closed. Since $f(x, y, z)$ is the gradient of $F(x, y, z) = xyz$, we conclude that the integral is zero! □