MA 1C RECITATION 05/14/15

1. Green's Theorem

Today we'll talk about Green's theorem, a powerful and fundamental result in calculus, and the first version of what I consider one of the most beautiful and elegant theorems in mathematics: the (general) Stokes' Theorem. I'll just write the formula here:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

Here Ω denotes some (usually compact) region, $\partial\Omega$ denotes its boundary taken with positive orientation, ω is a differential form (e.g. something like dx), and d is a certain map between functions called the *exterior* derivative.

This shouldn't make sense to you now, but the danger in this part of the course is learning things like Green's Theorem and Stokes' Theorem and only seeing a random list of crazy theorems and not being able to see the unifying theme behind them. Actually, even the general Stokes' theorem is a generalization of the Fundamental Theorem of Calculus. However, know that there is in fact a grand theory behind it all, and that by understanding the proper context, you can sort these results in your mind to make them easier to remember.

But we will just focus on applications of Green's Theorem today. Before we begin, we need to recall some terminology.

Definition 1.1. A **simple closed curve** $\gamma : [a, b] \to \mathbf{R}^n$ is a function such that

- $\gamma(a) = \gamma(b)$
- γ has finite length, and
- γ does not intersect itself, that is, if $x, y \in [a, b]$ are such that $x \neq y$, then $\gamma(x) = \gamma(y)$ if and only if $x = a$ and $y = b$, or vice versa.

If these land in $\mathbf{R}^2$, that is, if we're looking at curves in the plane (as we usually do in this course), these are often called **Jordan curves**.

Theorem 1.2 (General Green's theorem)**.** *Suppose that R is some closed, bounded, path-connected region in $\mathbf{R}^2$, such that the boundary of R is formed by the curves $C_1, \ldots, C_n$, where C_1 goes around the outside of R, the curves $C_2, \ldots, C_n$ all lie within C_1, and all of these curves are oriented in the counterclockwise direction. Now, suppose that $P, Q : \mathbf{R}^2 \to \mathbf{R}$ are a pair of maps with continuous partial derivatives in an open neighborhood of R. Then, we have the following equality:*

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \left(\oint_{C_1} P\, dx + Q\, dy \right) - \sum_{i=2}^{n} \left(\oint_{C_i} P\, dx + Q\, dy \right)$$

However, we will often be in a much simpler situation, where we have a region with no holes in it, like some closed ball. Here we can use the following special case of the theorem above, which is what people usually mean when they say "Green's theorem."

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Theorem 1.3 (Green's Theorem, special case)**.** *Keeping the notation as above, we have*

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_{C_1} P\, dx + Q\, dy.$$

2. Three Applications of Green's Theorem

So what does Green's Theorem do? We have a pair of functions P and Q and sends an integral involving them involving their partials $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$. Also, it transforms a line integral over some curve C into an integral over some region R. This suggests that we might want to use Green's theorem in the following three situations.

(a) **The curve is bad.** If we are trying to integrate a pair of functions over a nasty curve, we might want to use Green's theorem to transform this integral into one over a region, where we might be lucky and see that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is zero or make the integral simpler.

(b) **The region is bad.** If we have a bad looking region, we might want to a line integral around it instead of trying to integrate over it directly.

(c) **Switch between line integrals.** Sometimes we can make integrating over our line integral easier by using Green's theorem. Suppose that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is identically zero on some region R. Then for any two simple closed curves C_1, C_2 in R with C_1 contained inside of C_2, we can apply Green's theorem to see that

$$0 = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \left(\oint_{C_1} P\, dx + Q\, dy \right) - \left(\oint_{C_2} P\, dx + Q\, dy \right)$$

in other words,

$$\oint_{C_1} P\, dx + Q\, dy = \oint_{C_2} P\, dx + Q\, dy.$$

Thus, we can switch our integral between two curves! For instance, if we have some crazy-looking zig-zagging curve but know that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv 0$, we can just integrate along a circle inside of the region, and get the same answer.

Let's see an example of each of these situations.

Example 1. Let $a, b \in \mathbf{R}$ be constants, and $n \in \mathbf{N}$. What is

$$\int_{C_n^+} a\, dx + b\, dy$$

where C_n^+ is a positively oriented (i.e. counterclockwise) n-gon with side length 1, centered at $(0, 0)$, and one vertex on the x-axis.

Solution. While you might be able to do this by hand for 3-gons (triangles) or 4-gons (squares), doing this in general seems pretty awful. But Green's theorem comes to the rescue! If R is the region enclosed by our n-gon, we have

$$\begin{aligned}\oint_{C_n^+} a\,dx + b\,dy &= \iint_R \left(\frac{\partial(b)}{\partial x} - \frac{\partial(a)}{\partial y}\right) dx\,dy \\ &= \iint_R (0-0)\,dx\,dy \\ &= 0.\end{aligned}$$

□

Note that this result also holds if we integrate along closed curve. Thus, recalling the theorem relating gradients and line integrals from last week, we know that the function $(x, y) \mapsto (a, b)$ has a potential, that is, can be written as a gradient of some function. Indeed, that function is $F(x, y) = ax + by$.

Example 2 (Area of an ellipse)**.** Find the area of the region in $\mathbf{R}^2$ contained with boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. The area of any region R can be given by the integral $\iint_R 1\,dx\,dy$. Thus, if we choose $P(x, y) = -y/2$ and $Q(x, y) = x/2$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, and thus that

$$\iint_R 1\,dx\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \frac{1}{2}\oint_{C^+} x\,dy - y\,dx$$

where C^+ is the boundary curve of our ellipse. We will parametrize C^+ by $\gamma : [0, 2\pi] \to \mathbf{R}^2$ where $\gamma(t) = (a\cos(t), b\sin(t))$. Thus, we have

$$\begin{aligned}\frac{1}{2}\oint_{C^+} x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} (-y, x)\Big|_{\gamma(t)} \cdot \gamma'(t)\,dt \\ &= \frac{1}{2}\int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab(\sin^2(t) + \cos^2(t))\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt \\ &= ab\pi.\end{aligned}$$

□

Note that our choice of P and Q were not the only possible choice above. If we choose, say $P(x, y) = 0$ and $Q(x, y) = x$, we would get the same answer. (Check it out!)

This is a part of integration where "art meets science." You've also experienced this when you have to decide, for instance, how to decompose your integrand when using integration by parts. While the theorems give you the framework, it takes a keen eye and experience doing such integrals to find, say, choices of P and Q that make the integral easiest to evaluate. This is not something to worry too much about for this class, since it's just an introduction to the subject and most of the integrals will be straightforward, but if you go into most technical fields, especially anything related to physics or engineering, developing the "art" side of integration is an invaluable skill.

Example 3. Let $f(x,y)$ be the following function from $\mathbf{R}^2 \to \mathbf{R}$:

$$f(x,y) = \begin{cases} \arctan(y/x), & x > 0 \\ \pi/2, & x = 0, y > 0 \\ -\pi/2, & x = 0, y < 0 \\ \arctan(y/x) + \pi, & x < 0, y > 0 \\ \arctan(y/x) - \pi, & x < 0, y < 0 \end{cases}$$

and let $F(x,y) = \nabla f(x,y)$. On $\mathbf{R}^2 - \{\text{negative } x\text{-axis}\}$, we have

$$F(x,y) = \nabla f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Suppose that we extend this function F to all of $\mathbf{R}^2 \backslash \{(0,0\}$. We can do this since F is defined everywhere that the denominator is nonzero.

Suppose that C_1^+ is the positively oriented curve that corresponds to a square of side length 1 centered at the origin, with sides parallel to the x and y axes. What is the integral of $F(x,y)$ over C_1^+?

Solution. To find this, we could try to just integrate around the square, but it would be a long, ugly calculation. Instead, if we write $F(x,y) = (P(x,y), Q(x,y))$, we note that

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{(x^2+y^2) - 2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = -\frac{(x^2+y^2) - 2y^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

and so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ Thus, we know that on any region R that doesn't contain a small ball around the origin, we have $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = 0$. Thus, we can use our curve-switching technique mentioned earlier. Therefore, for any curve C_2^+ contained entirely within our square C_1^+, we have $\oint_{C_1^+} Pdx + Qdy = \oint_{C_2^+} Pdx + Qdy$. So let's choose a particularly easy curve to work with, like the circle of of radius 1/3 around the origin, which we parametrize as $\gamma(t) = (\frac{1}{3}\cos(t), \frac{1}{3}\sin(t))$. Then

$$\begin{aligned}
\oint_{C_1^+} P\,dx + Q\,dy &= \oint_{C_2^+} P\,dx + Q\,dy \\
&= \int_0^{2\pi} \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\Big|_{\gamma(t)} \cdot \gamma'(t)\,dt \\
&= \int_0^{2\pi} \left(-\frac{\frac{1}{3}\sin(t)}{\frac{1}{9}(\sin^2(t)+\cos^2(t))}, \frac{\frac{1}{3}\cos(t)}{\frac{1}{9}(\sin^2(t)+\cos^2(t))}\right) \cdot \left(-\frac{1}{3}\sin(t), \frac{1}{3}\cos(t)\right)\,dt \\
&= \int_0^{2\pi} 1\,dt \\
&= 2\pi.
\end{aligned}$$

□

Here's another application of the general Green's theorem.

Example 4. What is the area of an annulus A with outer radius r and inner radius s? We know the answer to this via elementary methods, but let's try and do this using Green's theorem.

Set $F = (-y, 0)$ and let C_r and C_s be circles of radius r and s respectively, traversed counterclockwise. Then

$$\begin{aligned}\int\int_A 1\,dx\,dy &= r^2\int_0^{2\pi}(-\sin t, 0)\cdot(-\sin t, \cos t)\,dt - s^2\int_0^{2\pi}(-\sin t, 0)\cdot(-\sin t, \cos t)\,dt \\ &= (r^2 - s^2)\int_0^{2\pi}\sin^2 t\,dt \\ &= (r^2 - s^2)\frac{1}{2}[x - \sin(x)\cos(x)]\Big|_0^{2\pi} \\ &= (r^2 - s^2)\pi.\end{aligned}$$

3. Divergence and Curl

This week's we're just introducing the concepts of divergence and curl, so we now have the complete set of new tools available for multivariable calculus (div, grad, and curl). The remainder of this course will essentially be combining these objects in clever ways and synthesizing it with what you already know about calculus to prove some amazing things.

We're not doing anything serious with these yet, but it's nice to at least get a little familiar with them so you're not scared when they get mentioned later on.

Definition 3.1. For a differentiable function $F : \mathbf{R}^3 \to \mathbf{R}^3$, the **divergence** of F is the *scalar field* $\mathbf{R}^3 \to \mathbf{R}$

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Remark 3.2. The notation $\nabla \cdot F$ is a clever mnemonic for the divergence. The way that I distinguish this from the similar mnemonic for curl is to remember that the "d" in "divergence" corresponds to "dot" product.

Physically, one way to think of the divergence is that it measures the "attraction" of a point in a particular way. There are some technical subtleties here, but this intuition sufficiently well for say, vector fields on surfaces. For instance, say you have a vector field on a surface. Pick a point on the surface. Suppose it has a lot of arrows going out from it, then it will have positive divergence. If it has a lot of arrows pointing into it, it will have negative divergence.

Definition 3.3. For a differentiable function $F : \mathbf{R}^3 \to \mathbf{R}^3$, the **curl** of F is the function $\mathbf{R}^3 \to \mathbf{R}^3$

$$\operatorname{curl}(F) = \nabla \times F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right)$$

The curl is sometimes written as the "determinant" of the matrix

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Remark 3.4. This "determinant formula" is why we have the notation $\nabla \times F$ for curl. Again, to keep this straight, the "c" in "curl" corresponds to "cross" product. Another way to remind yourself is taking the determinant above requires "curling" around the matrix to get the coefficients.

Physically, the curl is supposed to measure the rotation of a 3D vector field.

Example 5. Suppose that $F : \mathbf{R}^3 \to \mathbf{R}^3$ is a C^2 function (i.e. a function whose partials exist at least up to deg. 2 and are continuous). What is the divergence of the curl of F?

Solution. By definition, we have

$$\operatorname{curl}(F) = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right)$$

Therefore, we have

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}(F)) &= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\
&= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} \right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \right) \\
&= \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y} \right) + \left(\frac{\partial F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x} \right) \\
&= 0.
\end{aligned}$$

Note that it is essential that F is C^2, so that the partials commute. □

As an easy exercise, try and show that the curl of any gradient is zero.