MA 1C RECITATION 05/21/15

1. Something Strange

As you move onto higher math, you often run into something called the "curse of dimensionality," which, roughly speaking, is the idea that your intuition for how things work in the 2 or 3 dimension fails horribly in higher dimensions. This is a recurring theme for things like 4-dimensional spaces and whatnot that you will see in higher geometry/topology and physics, but it is also a prevalent in fields like data mining, where you often think of data obeying certain parameters as lying in some n-dimensional space.

We'll get our first taste of this in our recitation today. Try and think about how to solve the following questions. If you can do these problems without any issue, you should be in good shape for any change of coordinates we'll encounter in this class.

(a) Show that the five-dimensional unit ball $B_5 = \{\mathbf{x} \in \mathbf{R}^5 : ||x|| \leq 1\}$ has volume $8\pi^2/15$.

(b) Show that this volume is the largest volume attained by any n-dimensional unit sphere. In other words, for any $B_n = \{\mathbf{x} \in \mathbf{R}^n : ||x|| \leq 1\}$, we have $\mathrm{vol}(B_n) < \mathrm{vol}(B_5)$ for $n \neq 5$.

Thus, we see that 5-dimension sphere take up "more space" than any of the other spheres in any other dimension! Our goal today is to prove this result by using the tools that we have developed so far.

2. Change of Variables

Like most things in multivariable calculus, it is best to understand what happens in the single variable case first. Recall the following result from Math 1a.

Theorem 2.1 (Change of variables, single variable case)**.** *Suppose that f is a continuous function over the interval $(g(a), g(b))$ and that g is a C^1 map from (a, b) to $(g(a), g(b))$. Then we have*

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f(g(x)) \cdot g'(x)\, dx.$$

Let's break this down a little. We see that the integral of f over the $(g(a), g(b))$ is the same as the integral of $f \circ g$ over the interval (a, b), except that we need to correct for how g "shifts the space" going into f. In single variables, we can represent this change via differential forms. Namely, we integrate by dx on the left, representing the change in x, and on the right, we integrate with respect to $d(g(x)) = g'(x)\, dx$.

We want to try and do something similar for multiple variables. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a scalar field and let $g : \mathbf{R}^n \to \mathbf{R}^n$ be a differentiable map. But how do we get a number that represent how a function "shifts space"? Well, we know that the the

Date: May 21, 2015.

Jacobian $D(g(\mathbf{x}))$ measures small changes in the vector $\mathbf{x}$, and we recall from Math 1b that $\det(D(g(\mathbf{x}))$ measures the volume of the unit cube under the map $D(g(\mathbf{x}))$. Thus, this quantity, the determinant of the Jacobian of g, tells us how g is shifting the space around $\mathbf{x}$! Indeed, we have the following result.

Theorem 2.2 (Multivariable Change of Variables)**.** *Suppose that R is an open region in $\boldsymbol{R}^n$, that g is a C^1 map $\boldsymbol{R}^n \to \boldsymbol{R}^n$ on an open neighborhood of R, and that f is a continuous function on an open neighborhood of the region $g(R)$. Then*

$$\int_{g(R)} f(\boldsymbol{x})\, dV = \int_R f(g(\boldsymbol{x})) \cdot \det D(g(\boldsymbol{x}))\, dV.$$

3. Common Applications of Change of Variables

There are three common variable changes that we do in multivariable calculus: polar coordinates, cylindrical coordinates, and spherical coordinates.

Theorem 3.1 (Polar Change of Variables)**.** *Let $\gamma : [0, \infty) \times [0, 2\pi)$ be the polar coordinate map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Note that γ is C^∞. Then $D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$, so $\det(D(\gamma(r, \theta))) = r$, we so*

$$\int_{\gamma(R)} f(x, y)\, dV = \int_R f(r\cos(\theta), r\sin(\theta)) \cdot r\, dV$$

for any region R in $\boldsymbol{R}^2$ and any continuous function f on an open neighborhood of R.

In other words, if we have a region R described by polar coordinates, we can say that the integral of f over $\gamma(R)$ is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over this region interpreted in Euclidean coordinates. For example, suppose that R was the unit disk, which we can express using our polar coordinates map as $\gamma([0, 1] \times [0, 2\pi))$. Then, change of variables tells us that the integral of f over the unit disk is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over the Euclidean coordinate rectangle $[0, 1] \times [0, 2\pi)$.

We can similarly describe cylindrical coordinates.

Theorem 3.2 (Cylindrical Change of Variables)**.** *Let $\gamma : [0, \infty) \times [0 \times 2\pi) \times \boldsymbol{R}$ be the cylindrical coordinate map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Now, γ is C^∞ and*

$$D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\det(D(\gamma(r, \theta))) = r$ and so

$$\int_{\gamma(R)} f(x, y, z)\, dV = \int_R f(r\cos(\theta), r\sin(\theta), z) \cdot r\, dV,$$

for any region R in $\boldsymbol{R}^3$ and any continuous function f on an open neighborhood of R.

Spherical coordinates are just a slightly complicated twist on this general theme.

Theorem 3.3 (Spherical Change of Variables)**.** *Let $\gamma : [0,\infty) \times [0,\pi) \times [0,2\pi)$ be the cylindrical coordinate map $(r,\phi,\theta) \mapsto (r\cos(\phi), r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta))$. Now γ is C^∞ and*

$$D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi)\cos(\theta) & r\cos(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & r\cos(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) \end{bmatrix}$$

so $\det(D(\gamma(r,\theta))) = r^2\sin(\phi)$ and we have

$$\int_{\gamma(R)} f(x,y,z)\,dV = \int_R f(r\cos(\phi), r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta)) \cdot r^2\sin(\phi)\,dV,$$

for any region R in $\boldsymbol{R}^3$ and any continuous function f on an open neighborhood of R.

Other common coordinate transformations. These are a bit simpler, so I will omit the details.

- Translations: $(x,y,z) \mapsto (x+c_1, y+c_2, z+c_3)$. The determinant of the Jacobian of such maps is 1. This is pretty obvious, but I state it for completeness.
- Scalings: e.g $(x_1,\ldots,x_n) \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n)$. The determinant of the Jacobian of such maps is the product of the scaling constants, that is, $\lambda_1 \cdots \lambda_n$.
- Various composition of these maps. By the chain rule, we know that the determinant of the Jacobian of the composition is just the product of the determinant of the Jacobians of the individual maps.

These things are pretty routine and straightforward. The only difficult part is deciding which coordinate change admits the simplest integral. To see this, it's probably best to work through some examples.

4. Examples

Example 1. Find the area in $\mathbf{R}^2$ contained inside the ellipse

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

using change of variables.

Solution. We saw this last week, but here's a different way to do it. We want to integrate 1 over the region R contained inside the ellipse. To see this, note that R is the image of the unit disk D under the scaling map $\gamma(x,y) = (ax, by)$. Therefore, by an application of change of variables, we have

$$\int_{\gamma(D)} 1\,dV = \int_D 1 \cdot \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} dV = \int_D ab\,dV.$$

Now, by using the polar coordinates for the unit disk D, we describe D as the image of the rectangle $[0,1] \times [0,2\pi]$ under the map $\alpha(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$, we can apply change of variables again to get

$$\int_{\gamma(D)} 1\,dV = \int_{[0,1]\times[0,2\pi]} 1 \cdot \det(D\alpha)\,dV = \int_0^1 \int_0^{2\pi} ab \cdot r\,d\theta\,dr = \pi ab.$$

Thus, the area of R is πab. □

Example 2. Find the area enclosed by the astroid curve $\{(x,y) : x^{2/3} + y^{2/3} = 1\}$.

Solution. This problem might look very familiar! However, it is likely that your solution to this was rather complicated. We'll see that change of coordinates will give us a slick solution.

From looking at the graph of this function, we might try to use polar coordinates, but if you do that, you will run into problems. Instead, based on the fact that we used polar coordinates $(\cos(\theta), \sin(\theta))$ to describe the points on the unit circle $x^2 + y^2 = 1$, we want to try and describe our equation via the parametrization $(\cos^3(\theta), \sin^3(\theta))$. Thus, we can express the region R contained within the curve as the image of the rectangle $[0, 1] \times [0, 2\pi]$ under the map

$$\gamma(r, \theta) = (r\cos^3(\theta), r\sin^3(\theta)).$$

Thus, by using change of variables with this map, we have

$$\begin{aligned}
\int_R 1\, dV &= \int_{[0,1]\times[0,2\pi]} 1 \cdot \det \begin{bmatrix} \cos^3(\theta) & -3r\cos^2(\theta)\sin(\theta) \\ \sin^3(\theta) & 3r\sin^2(\theta)\cos(\theta) \end{bmatrix} dV \\
&= \int_0^1 \int_0^{2\pi} 3r(\cos^4(\theta)\sin^2(\theta) + \sin^4(\theta)\cos^2(\theta))\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} 3r\cos^2(\theta)\sin^2(\theta)(\cos^2(\theta) + \sin^2(\theta))\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} 3r\cos^2(\theta)\sin^2(\theta)\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} 3r\frac{\sin^2(2\theta)}{4}\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} 3r\frac{1-\cos(4\theta)}{8}\, d\theta\, dr \\
&= \int_0^1 \frac{3r\pi}{4}\, dr \\
&= 3\pi/8.
\end{aligned}$$

□

Example 3. We end our examples with one of the most famous calculations in mathematics: the calculation of the "Gaussian integral"

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

I think this way of solving this integral is originally due to Laplace, from the late 18th century.

Solution. Write $I = \int_{-\infty}^{\infty} e^{-x^2}$. We have

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy.$$

All we've done is change the dummy variable for the second integral. We can express this quantity as a double integral

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$

We can express this in polar coordinates, so we have

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta.$$

The integrand is independent of π, so we get

$$I^2 = 2\pi \int_0^\infty e^{-r^2}\, r\, dr\, d\theta.$$

By substituting $u = r^2$, so $du = 2r\, dr$, we see that

$$\int_0^\infty e^{-r^2}\, r\, dr = \frac{1}{2}\int_0^\infty e^{-u}\, du = \frac{1}{2}.$$

Thus, $I^2 = 2\pi \cdot \frac{1}{2} = \pi$ and so $I = \sqrt{\pi}$. □

Remark 4.1. This is essentially the reason why π appears in many probabilistic and statistical formulas. The ubiquity of Gaussian-style integrals in the world is quite mysterious, but one interpretation that I like, is that the reason for this is because *the universe is inherently quantum-mechanical, and thus probabilistic.*

How does this view approach things like classical mechanics, which is not probabilistic (at least, the way we usually learn it)? It says that these definite things that we perceive are manifestations of *the law of large numbers*, which ensures that averages occur much more frequently than other outcomes. This law is at work, for instance, when a balloon floats through the air. Inherently, air molecules are hitting the balloon randomly in all directions, but what we *perceive* is balloon smoothly floating through the air in a distinct trajectory, as opposed to randomly jumping about.

5. The Solution

Now, let's use the change of variables to prove our surprising result. We can generalize our three-dimensional spherical coordinates to n-dimensional spherical coordinates. In other words, let $r \in [0, \infty)$, $\phi_1, \dots, \phi_{n-2} \in [0, \pi)$, and $\theta \in [0, 2\pi)$. Now, consider the map γ that sends a point $(r, \phi_1, \dots, \phi_{n-2}, \theta)$ to a point in $\mathbf{R}^n$ with the coordinates

$$\begin{aligned}
x_1 &= r\cos(\phi_1)\\
x_2 &= r\sin(\phi_1)\cos(\phi_2)\\
x_3 &= r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3)\\
&\vdots\\
x_{n-2} &= r\sin(\phi_1)\cdots\sin(\phi_{n-3})\cos(\phi_{n-2})\\
x_{n-1} &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-2})\cos(\theta)\\
x_n &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-2})\sin(\theta).
\end{aligned}$$

We can show that this point $\mathbf{x}$ is a point that is distance r from the origin that has angle ϕ_i with the first n-2 coordinate axes and angle θ with the $(n-1)$-th axis. Thus, we see that this is just a generalization of spherical coordinates to n dimensions.

Inductively, we can show that

$$\det(D(\gamma)) = r^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})$$

and thus the volume of the n-dimensional ball, via change of variables, is just the integral

$$\int_0^1 \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} r^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2})\, d\phi_1\, d\phi_2 \,\cdots\, d\phi_{n-1}\, dr.$$

From there, we can use induction to prove the recursion relation

$$\mathrm{vol}(B_n) = \frac{2\pi}{n} \mathrm{vol}(B_{n-2}),$$

which tells us that for $n \geq 7$, $\mathrm{vol}(B_n)$ is strictly smaller than $\mathrm{vol}(B_{n-2})$. Checking the volumes for the balls $B_1, \ldots, B_6$ then shows that $\mathrm{vol}(B_5)$ is the greatest amongst those six balls: therefore, the volume of the five-dimensional unit ball B_5 is greater than the n-dimensional volume of **any** of the other n-dimensional unit balls, because the volumes are (as shown) decreasing for $n > 6$!

These calculations are a little lengthy, but they are totally doable! So if you're bored over the long weekend and want some practice, try them out! (Feel free to ask me if you get stuck.)