MA 1C RECITATION 06/07/12

1. Administrivia

The final exams should be available on Friday after class. You have 4.5 hours to due them and they are due **Wednesday, June 13 at noon** in the *Ma 1c slots for your section* (where you usually submit your homework on Mondays). Do NOT put them in my mailbox, Prof. Flach mailbox, the return slots, slip under the door to our offices, etc.

It follows the standard testing format: open book, class notes, TA notes, your own notes, posted solutions, but you can't use the internet. You are allowed to use a computer, but only for basic arithmetic, for plotting, or to work out *single-variable* integrals so that you're not stuck at some tricky integral. However, no symbolic solving, and no solving of multivariate integrals. You still have to do *some* integrals...this is a calculus class, after all!

Make sure that you show your work, not only because it is the only way to get full credit, but if you just write the answer and the answer is wrong, you cannot even get partial credit. Make sure that you use a blue book, and double check that you put your name and your section on there so that it can be graded and returned as promptly as possible.

In addition to the notes that I have posted, Alden also has a set of notes on his website that you can use. He often has a different take on things than I do, so it's certainly worth checking out what he thinks is important.

The final review for this class will be **8pm Sunday in Sloan 151**. Also, the practical section has a final review **noon Saturday** in the same location. They cover essentially the same material, so if you want a refresher from a slightly different perspective, or think that the Sunday review is too late, you can check it out.

Best of luck!

2. Final Exam Review Examples

Since there will be a final review, I will go over some carefully chosen examples that address some common errors. The theme behind today's recitation is **there's usually a clever way to do it**, meaning that there are often techniques aside from the standard approach that you can use to solve problems quickly. These are good ways to save time on the exam, so that if you run through the problems and have some extra time, you can try and redo the problem the obvious and straightforward way to be doubly sure of your answer.

Example 1. Calculate the surface area of a torus around the circle $x^2 + y^2 = R^2$ with internal radius r. (The internal radius being the radius of the circle that you rotate around the z-axis to get the torus.)

Solution. From Apostol, we know that the torus is parametrized by the map $\phi : [0, 2\pi] \times [0, 2\pi] \to \mathbf{R}^3$ with

$$\phi(u, v) = (\cos(u)(R + r\cos(v)), \sin(u)(R + r\cos(v)), r\sin(v)).$$

Date: June 7, 2012.

One approach to finding the surface area is simply to take the surface integral:

$$\begin{aligned}
\iint_T 1\, dT &= \iint_{[0,2\pi]^2} \left|\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right|\right| \\
&= \iint_{[0,2\pi]^2} ||(-\sin(u)(R + r\cos(v)), \cos(u)(R + r\cos(v)), 0) \times (-r\cos(u)\sin(v), -r\sin(u)\sin(v), r\cos(v))||\, du\, dv \\
&= \iint_{[0,2\pi]^2} ||(r\cos(u)\cos(v)(R + r\cos(v)), r\sin(u)\cos(v)(R + r\cos(v)), -r\sin(v)(R + r\cos(v)))||\, du\, dv \\
&= \iint_{[0,2\pi]^2} \sqrt{(r\cos(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(v)(R + r\cos(v)))^2}\, du\, dv \\
&= \iint_{[0,2\pi]^2} r(R + r\cos(v)) \cdot \sqrt{\cos^2(u)\cos^2(v) + \sin^2(u)\cos^2(v) + \sin^2(v)}\, du\, dv \\
&= \iint_{[0,2\pi]^2} r(R + r\cos(v)) \cdot \sqrt{\cos^2(v) + \sin^2(v)}\, du\, dv \\
&= \iint_{[0,2\pi]^2} R \cdot r + r^2\cos(v)\, du\, dv \\
&= 4\pi^2 Rr.
\end{aligned}$$

Another approach, the clever one, is to apply Pappus's theorem for surface area. We can regard our torus as the surface of revolution given by revolving the curve $(x - R)^2 + z^2 = r^2$ around the z-axis. the length of such a circle is $2\pi r$, and the center of mass a circle is obviously its center, which is at $(R, 0, 0)$. Therefore, the distance of this circle's center of mass from the z-axis is R, so Pappus's theorem tels us that the area of the torus T is $2\pi R \cdot 2\pi r = 4\pi^2 Rr$, which agrees with our answer above. □

Example 2. Let $S = \{(x, y, z) : x^2+y^2+z^2 = 1, x, y, z \geq 0\}$ be an octant of the unit sphere and let $C^+ = \partial S$ be the boundary of S traversed in the counterclockwise direction viewed from the positive z-axis. if $F(x, yz) = (x^4, y^4, z^4)$, what is $\int_C F \cdot dC$?

Solution. One approach here is to just take line integrals. Namely, we can parametrize C as three curves

$$\begin{aligned}
\gamma_1(t) &= (\cos(t), \sin(t), 0) \\
\gamma_2(t) &= (0, \cos(t), \sin(t)) \\
\gamma_3(t) &= (\sin(t), 0, \cos(t))
\end{aligned}$$

where $t \in [0, \pi/2]$. Note that these traverse the curve in the counterclockwise direction. Therefore, we have

$$\begin{aligned}
\int_C F \cdot dC &= \sum_{i=1}^{3} \int_0^{\pi/2} (F \circ \gamma_i(t)) \cdot (\gamma'(t))\, dt \\
&= \sum_{i=1}^{3} \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t)\, dt \\
&= 3\int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t)\, dt \\
&= \left[-3\int_0^{\pi/2} \cos^4(t)\sin(t)\, dt\right] + \left[3\int_0^{\pi/2} \sin^4(t)\cos(t)\, dt\right].
\end{aligned}$$

To evaluate these last two integrals we u-substitute $u = \cos(t)$ for the first integral and $u = \sin(t)$ for the latter integral, so that we obtain

$$\begin{aligned}
\int_C F \cdot dC &= \left[3\int_1^0 u^4\, du\right] + \left[3\int_0^1 u^4\, du\right] \\
&= -3\int_0^1 u^4\, du + 3\int_0^1 u^4\, du \\
&= 0.
\end{aligned}$$

Alternatively, we could have used Strokes' theorem, which tells us the integral of F over C is the integral of $(\nabla \times F) \cdot \mathbf{n}$ over S. However, we have

$$\begin{aligned}
\operatorname{curl}(F) &= \nabla \times F \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= (0-0, 0-0, 0-0) \\
&= (0,0,0).
\end{aligned}$$

Therefore, $(\nabla \times F) \cdot \mathbf{n}$ must be zero and so the integral of it over S must be zero. □

Example 3. Find the area of the region R enclosed by the curve

$$\gamma(t) = (\cos(t), \sin(3t))$$

where $t \in [0, 2\pi]$.

Remark 2.1. Such a curve is called a *Lissajous curve* and originated in the theory of complex harmonic motion. One reason it's interesting is because if you shift how fast you're traversing across the cosine or sine part, say by having $\cos(12t)$ in x-coordinate, you end up with a dramatically different looking curve. They also often appear as "slices" of higher-dimensional objects that correspond to some natural physical motion.

Another way that there objects show up in mathematics is as the projection of a (mathematical) knot[1] from 3-space to the plane. Studying the "shadows" of higher dimensional objects in this way is a common technique in modern mathematics and physics.

Solution. Okay, we're in a plane, asked to take an area of something enclosed by a curve which we don't even have to go through the trouble to parametrize. It's even oriented clockwise! Your first instinct should be to use Green's theorem. We have

$$\operatorname{area}(R) = \iint_R 1\, dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\, d\gamma.$$

Thus, we have

$$\begin{aligned}
\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\, d\gamma &= \int_0^{2\pi} \left(-\frac{\sin(3t)}{2}, \frac{\cos(t)}{2}\right) \cdot (-\sin(t), 3\cos(3t))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \sin(3t)\sin(t) + \cos(3t)\cos(t)\, dt.
\end{aligned}$$

OK, so we have a slightly tricky integral. How do we solve this. Trig identities! But which ones? The triple-angle formulas, of course!

$$\begin{aligned}
\cos(3t) &= 4\cos^3(t) - 3\cos(t) \\
\sin(3t) &= 3\sin(t) - 4\sin^3(t).
\end{aligned}$$

[1]same as the knots of string you know and love, but by attaching the two pieces together at the ends

(We don't actually expect you to know these off the top of your head. This is why you're allowed to use computers for single variable integrals.) Therefore, we have

$$\begin{aligned}
\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right)\, d\gamma &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^4(t) - \sin^4(t))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t)(1-\sin^2(t)) - \sin^2(t)(1-\cos^2(t)))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t) + \sin^2(t)\cos^2(t) - \sin^2(t)\cos^2(t))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t))\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \cos^2(t) - \sin^2(t)\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \cos(2t)\, dt \\
&= 0.
\end{aligned}$$

So the answer is zero! Wait, that can't be right. By inspection, we see that the region inside our curve is clearly nonzero. Where did we go wrong? It was in the application of Green's theorem. Recall that the hypothesis of Green's theorem states that it only applies to simple closed curves. The curve γ is closed, but it is *not* simple, because the curve intersects itself. So how do we solve this? We need to break our curve into parts and apply Green's theorem to each component.

Right: If we restrict parameter t of γ to $[-\pi/3, \pi/3]$, we get the right-most part of the curve. Here, γ is oriented counterclockwise, so we can find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} \cos(2t)\, dt = \left.\frac{\sin(2t)}{4}\right|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{4}.$$

Left: By restricting the parameter t to $[2\pi/3, 4\pi/3]$, we obtain the left-most part of the curve. Here, γ is also oriented counterclockwise, so we can once again find the area by evaluating the integral

$$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} \cos(2t)\, dt = \left.\frac{\sin(2t)}{4}\right|_{2\pi/3}^{4\pi/3} = \frac{\sqrt{3}}{4}.$$

Center: Finally, by restricting the parameter t to $[\pi/3, 2\pi/3] \cup [4\pi/3, 5\pi/3]$, we get the center piece, but here γ is oriented clockwise, so we need to reverse the orientation. We get

$$\frac{1}{2}\int_{2\pi/3}^{\pi/3} \cos(2t)\, dt + \frac{1}{2}\int_{5\pi/3}^{4\pi/3} \cos(2t)\, dt = \frac{\sqrt{3}}{2}.$$

Note that the curve γ here was defined piecewise. This is fine. It still a simple closed curve that is counterclockwise oriented. We just need to break the integral into two parts.

Thus, by summing these areas together, we see that the area of the region R enclosed by the curve γ is $\sqrt{3}$. $\square$