MA 1C RECITATION 05/03/12

1. Announcements

Midterms are available and due on Monday at noon. There will be a review session today from 8-9pm in Sloan 151.

Read Alden's class email about critical points and things like checking for compactness when doing optimization problems. It's long, but it's important. For a simple example of this importance, just consider trying to find the extrema of the function $f(x) = x$ on $(0, 1)$. What are the extrema of f? You might say they are 0 or 1, but they are not even in the domain! Thus, we see that compactness is crucial for finding optimal values. For instance, if we considered $[0, 1]$ instead of $(0, 1)$ above, we would get our extrema.

2. Midterm Review Examples

There are a lot of places where you can get a general overview of things: class notes, Alden's notes, the textbook, most other recitations. Instead, we're going to try and work through some thorny problems that will hopefully refresh your memory and more importantly, remind you to check certain things so that you don't make common mistakes on the exam.

A couple of these are deliberately trickier than things you might see on a midterm, so don't worry too much if you cannot do them all quickly. However, if you do manage to do all of these without any issue, you should be in good shape for the exam!

Example 1. Consider the set of all numbers of the form $2^{-n} + 5^{-m}$ for all $m, n = 1, 2, 3, \ldots$. What kind of set is this? (open, closed, bounded, unbounded, compact, noncompact?)

Solution. We will write the set

$$\begin{aligned}
S := \{2^{-n} + 5^{-m} \mid m, n = 1, 2, \ldots\} &= \bigcup_{m=1}^{\infty} \left\{ \frac{1}{2} + 5^{-m}, \frac{1}{4} + 5^{-m}, \ldots, \frac{1}{2^n} + 5^{-m}, \ldots \right\} \\
&= \left\{ \frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{5}^2, \ldots, \frac{1}{2} + \frac{1}{5^m}, \ldots \right\} \\
&\cup \left\{ \frac{1}{4} + \frac{1}{5}, \frac{1}{4} + \frac{1}{5^2}, \ldots, \frac{1}{4} + \frac{1}{5^m}, \ldots \right\} \\
&\cup \cdots \\
&\cup \left\{ \frac{1}{2^n} + \frac{1}{5}, \frac{1}{2^n} + \frac{1}{5^2}, \ldots, \frac{1}{2^n} + \frac{1}{5^m}, \ldots \right\} \cup \cdots
\end{aligned}$$

Now, S is not closed since it does not contain 0, which is a limit point of S. Furthermore, since $\frac{1}{2} + \frac{1}{5} \in S$, but $B(\frac{1}{2} + \frac{1}{5}, r)$ is not contained in S for any $r > 0$, we know that S is not open either. This set is bounded, e.g. because $S \subset [0, 1]$.

Date: May 3, 2012.

By Heine-Borel, a set in $\mathbf{R}^n$ is compact if and only if it is closed and bounded, so S is not compact. $\square$

Example 2. Let $S = GL(2, \mathbf{R})$ be the set of 2×2 invertible matrices with coefficients in $\mathbf{R}$. The standard topology that we put on S is to consider S as a subset of $\mathbf{R}^4$ (one for each coordinate of the matrix). Can you construct a closed set in S that covers the subset $S(3)$ of matrices of determinant 3?

Proof. It turns out that you can! Consider the function $\det : S \to \mathbf{R}$ that sends a matrix to its determinant. This is a continuous function! (How come?) Therefore, $\det^{-1}(\{3\})$ is a closed set that covers $S(3)$. (Why? Work it out, using the fact that the preimage of an open set of an continuous function is open) Indeed, it is exactly $S(3)$! $\square$

Example 3. An ant is crawling on a plane. The temperature at a point (x, y) is given by

$$T(x, y) = 70 + 5\sin(x)\cos(y)$$

in degrees. The ant's position at time t (say seconds) is $\gamma(t) = (t\cos(\pi t), t\sin(\pi t))$. How fast is the temperature under the ant changing at $t = 5$?

Solution. We need to compute the derivative of $T \circ \gamma(t)$ at $t = 5$. We compute

$$\begin{aligned}
\gamma(t) &= (5\cos(5\pi), 5\sin(5\pi)) = (-5, 0) \\
D\gamma &= \begin{bmatrix} \cos(\pi t) - \pi t\sin(\pi t) \\ \sin(\pi t) + \pi t\cos(\pi t) \end{bmatrix} \\
D\gamma(5) &= \begin{bmatrix} -1 \\ -5\pi \end{bmatrix} \\
DT &= (5\cos(x)\cos(y), -5\sin(x)\sin(y)) \\
DT(-5, 0) &= (5\cos(5), 0).
\end{aligned}$$

So by the chain rule, we have

$$[D(T \circ \gamma)] = (5\cos(5), 0)\begin{bmatrix} -1 \\ -5\pi \end{bmatrix} = -5\cos(5).$$

Hence, the temperature under the ant is changing at $-5\cos(5)$ degrees per second at $t = 5$. $\square$

Remark 2.1. For the following extrema questions, I will try and do them and set them up in slightly different ways from what has been done in previous recitations, notes, the book, solution, etc. Do not be disturbed. The underlying methodology is the same, even if it looks slightly different. (For instance, in the next example, I am obviously computing the gradient, but there is no gradient or ∇ anywhere in the solution.) However, the idea is that (1) you will do the problems yourself in your own way before you look at the strange solutions and (2) that relating each of these "bizarre" versions to the one in your head will help solidify these concepts, and maybe seeing the same technique from many different perspectives will finally make it "click" for you.

Example 4. Let

$$f(x, y) = x^3 - 3xy^2 + 2y^3.$$

Find and classify the extrema.

Solution. Any local extrema occur where both partial derivatives vanish. We have

$$\frac{\partial f}{\partial x} = 3x^3 - 3y^2$$
$$\frac{\partial f}{\partial y} = -6xy + 6y^2.$$

The first partial derivative is zero when $y = x$ or when $y = -x$. The second is zero when $y = x$ or $y = 0$. Therefore, all the local extrema lie on the line $y = x$, where

$$f(x, x) = x^3 - 3x^3 + 2x^3 = 0.$$

Hmm... that's a bit strange. What's going on?

Well, if we factor $f(x, y)$ we see what's up:

$$f(x, y) = (x - y)^2(x + 2y).$$

This is at least 0 when $x, y > 0$ and at most 0 when $x, y < 0$. It follows that if $y = x > 0$, then we have a relative minimum. If $y = x < 0$, then we have a relative maximum, and if $y = x = 0$, then we have neither a relative maximum nor a relative minimum because there are points arbitrarily close to $(0, 0)$ for which f is positive and points arbitrarily close to $(0, 0)$ for which f is negative. □

Example 5. Find the closest point to $(1, 1, 1)$ in the set $S = \{(x, y, z) \mid x^2 + y^2 = 4, z = 2\}$.

Solution. We want to minimize the function $f(x, y, z) = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$ that gives us the distance of a point from $(1, 1, 1)$. However, the distance function will reach its minimum and maximum at the same points as the distance squares, so it is enough to try and minimize the function

$$f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2.$$

Now, what constraints will give us our set S? Well, S is just a circle of radius 2 around the z-axis in the $z = 2$ plane. Therefore, we can write S as the intersection of the level sets

$$g_1(x, y, z) = x^2 + y^2 - z^2 = 0$$
$$g_2(x, y, z) = z = 2$$

which gives us our constraint. (Why did I write it in this way? Because the zeros of g_1 gives us a (double) cone from the origin, and g_2 gives us a plane, and by intersecting these level sets, we can visualize the solution.) Note that the level sets $g_1^{-1}(1)$ and $g_2^{-1}(2)$ giving us these points are closed, and the intersection of two closed sets in closed, so S is closed. We also see that S is bounded, so S is compact and thus f will achieve its extrema on S.

Now that we've phrased our optimization problem in the language of Lagrange multipliers, let's work through it! We want to minimize f with respect to the two constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 2$. To do this, we must first check that $\nabla(g_1)$ and $\nabla(g_2)$ are linearly independent for any (x, y, z). We compute

$$\nabla(g_1) = (2x, 2y, -2z)$$
$$\nabla(g_2) = (0, 0, 1).$$

Since we must have $x^2 + y^2 = 4$ by our constraint, we cannot have $x = 0$ and $y = 0$ at the same time, so there are no points in S such that these two vectors above are linearly dependent. Thus, we can indeed apply the method of Lagrange multipliers to find extrema that obey the constraints.

Now, we are looking for (x, y, z) with constants λ_1, λ_2 such that

$$\begin{aligned} \nabla(f) &= \lambda_1 \nabla(g_1) + \lambda_2 \nabla(g_2) \\ (2x-2, 2y-2, 2z-2) &= \lambda_1(2x, 2y, -2z) + \lambda_2(0, 0, 1). \end{aligned}$$

Therefore, we have the system of equations

$$\begin{aligned} 2x - 2 &= 2\lambda_1 x \\ 2y - 2 &= 2\lambda_1 y \\ 2z - 2 &= -2\lambda_1 + 2\lambda_2. \end{aligned}$$

If we multiply the equations $2x-2 = 2\lambda_1 x$ and $2y-2 = 2\lambda_y$ by y and x respectively, we obtain

$$2\lambda_1 xy = 2xy - 2x = 2xy - 2y$$

that is, $x = y$. Since $z = 2$, and because $x^2+y^2 = 4$, we see that we have checkpoints at $(-\sqrt{2}, -\sqrt{2}, 2)$ and $(\sqrt{2}, \sqrt{2}, 2)$.

It is enough to classify these points. Since S is compact, f must attain its absolute extrema on this set, and not only that, they must be one of our checkpoints! So we just plug these points into f. We then see that $(\sqrt{2}, \sqrt{2}, 2)$ is the closest point on S to $(1, 1, 1)$. □

The next solution uses Lagrange multipliers, but the first step is given in the language of differential forms, the way that you deal with integration in modern mathematics and physics. However, it's the same Lagrange multiplier setup you know and love, just expressed differently. As you can see, the solution steps are the same.

Example 6. Find the extrema of

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints

$$\begin{aligned} (x-1)^2 + (y-2)^2 + (z-3)^2 &= 9 \\ x - 2z &= 0. \end{aligned}$$

Proof. To solve this, we introduce two Lagrange multipliers and set up our differential equation:

$$2x\,dx + 2y\,dy + 2z\,dz = \lambda_1[2(x-1)\,dx + 2(y-2)\,dy + 2(z-3)\,dz] + \lambda_2(dx - 2\,dz).$$

This gives us the additional equations

$$\begin{aligned} 2x &= 2(x-1)\lambda_1 + \lambda_2 \\ y &= (y-2)\lambda_1 \\ z &= (z-3)\lambda_1 - \lambda_2. \end{aligned}$$

Adding the first and third equations and then using the second constraint to substitute $2z$ for x, we get

$$5z = (5z - 5)\lambda_1,$$

so we have

$$z = \frac{-\lambda_1}{1-\lambda_1}$$
$$y = \frac{-2\lambda_1}{1-\lambda_1},$$
$$x = \frac{-2\lambda_1}{1-\lambda_1}.$$

Substituting into the first constraint, we can solve for λ_1:

$$\lambda_1 = -\frac{1}{2} \text{ or} \frac{5}{2},$$

and so we have two checkpoints:

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \text{ and } \left(\frac{10}{3}, \frac{10}{3}, \frac{5}{3}\right)$$

We have an absolute minimum of 1 at the first point, and an absolute minimum of 25 at the second. $\square$

Warning 2.2. Uh oh! The "solution" above is deficient. What steps did I forget?

Example 7. Consider the function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(\mathbf{x}) = \mathbf{x}^T \cdot A \cdot \mathbf{x}$$

where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 2 & \cdots & 2 \\ 2 & 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 1. \end{bmatrix}$$

(a) Find the directional derivative of f at $(1, 1, \ldots, 1)$ in the direction $(1, -1, 1, -1, \ldots)$, where the 1 and -1 are alternating.
(b) Find and classify the critical points of f.

Solution. Let's first write $f(\mathbf{x})$ in a form that is easier to understand so we can apply our analysis. Let us write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, so we have

$$f(x_1, \ldots, x_n) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 2 & \cdots & 2 \\ 2 & 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 1. \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 + \sum_{i \neq 1} 2x_i \\ x_2 + \sum_{i \neq 2} 2x_i \\ \vdots \\ x_n + \sum_{i \neq n} 2x_i \end{bmatrix}$$
$$= \left(\sum_{i=1}^{n} x_i^2\right) + \left(\sum_{i} \sum_{\substack{j \\ i \neq j}} 4x_i x_j\right).$$

Therefore, the gradient of f is given by the vector

$$\nabla(f) = \left(2x_1 + \sum_{i\neq 1} 4x_i, 2x_2 + \sum_{i\neq 2} 4x_i, \dots\right).$$

Thus, at $(1, \dots, 1)$, we have

$$\nabla(f)(1, 1, \dots, 1) = (4n - 2, 4n - 2, \dots, 4n - 2).$$

Now, we want to take the directional derivative in the direction $(1, -1, \dots)$, but remember that we need to normalize the unit first! Thus, we have

$$\nabla(f)_{(1,-1,\dots)/||(1,-1,\dots)||}(1, 1, \dots, 1) = \frac{(4n-2) - (4n-2) + (4n-2) - (4n-2) + \cdots}{\sqrt[n]{n}}.$$

Hence, if n is even, we know that the value above is 0. If n is odd, we have $(4n - 2)/\sqrt[n]{n}$. $\diamond$

We now want to find the critical points of f. Look at the gradient and note that $\nabla(f) = (0, \dots, 0)$ if and only if $2x_i + \sum_{j\neq i} 4x_j = 0$ for every i. Observe that this condition gives us a system of linear equations, so we can represent this in matrix form as

$$\begin{bmatrix} 2 & 4 & 4 & \cdots & 4 \\ 4 & 2 & 4 & \cdots & 4 \\ 4 & 4 & 2 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 2. \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

The matrix is rank n, so by the Grand Theorem of Linear Algebra, we know many facts about it, but most importantly, its nullspace is zero dimensional. Therefore the only solution to this system is the trivial vector $\overline{x} = (0, 0, \dots, 0)$.

Hence, f has only one critical point: the origin. What kind of critical point is this?

To do this, consider the Hessian of f, which we calculate (by differentiating our entries in our gradient) to be

$$H_f = \begin{bmatrix} 2 & 4 & 4 & \cdots & 4 \\ 4 & 2 & 4 & \cdots & 4 \\ 4 & 4 & 2 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 2. \end{bmatrix}$$

What are the eigenvalues of this matrix? That is, for what values of λ does the matrix

$$\begin{bmatrix} 2-\lambda & 4 & 4 & \cdots & 4 \\ 4 & 2-\lambda & 4 & \cdots & 4 \\ 4 & 4 & 2-\lambda & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 2-\lambda. \end{bmatrix}$$

not have full rank?

The first guess is $\lambda = -2$, which would result in a matrix of all 4's, which has rank 1. Also, if $\lambda = 4n - 2$, we see that our matrix will be

$$\begin{bmatrix} 4-4n & 4 & 4 & \cdots & 4 \\ 4 & 4-4n & 4 & \cdots & 4 \\ 4 & 4 & 4-4n & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 4-4n. \end{bmatrix}$$

If we add all the rows of the matrix together, we can the zero vector, so the rows are not linearly independent and so the matrix does not have full rank. Hence, $4n - 2$ is also an eigenvector of our matrix. Therefore, we see that the Hessian has both positive and negative eigenvectors at $(0, \dots, 0)$ and so must be a saddle point. $\square$