MA 1C RECITATION 05/10/12

1. Line Integrals

Now that we've spent all this time building up the foundations of the theory, we can get to some awesome nontrivial applications. Most of what we do today will be computational and relatively straightforward.

Definition 1.1. For a function $f : \mathbf{R}^m \to \mathbf{R}^n$ and a continuous path $\gamma : [a, b] \to \mathbf{R}^m$, we define the **line integral of f along γ** to be

$$\int_\gamma f \, d\gamma := \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt.$$

While the line integral might seem to depend on the path, the following result implies that the integral only depend on the curve traced out by the line itself, and not a particular parametrization.

Theorem 1.2. *Suppose that $\gamma : [a, b] \to \boldsymbol{R}^n$ and $\alpha : [c, d] \to \boldsymbol{R}^n$ are two paths such that (1) α and γ have the same image in $\boldsymbol{R}^n$, (2) $\alpha(a) = \gamma(c)$, and (3) both paths traverse their images with the same orientation (roughly, move in the "same direction"). Then for any function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$, we have*

$$\int_\gamma f \cdot d\gamma = \int_\alpha f \cdot d\alpha,$$

provided that either integral exists.

Remark 1.3. This "parametrization-independent" theorem is the pedagogical reason why these are called *line* integrals (or contour integrals). Some books sometimes call these path integrals, but you should probably avoid this language, if only because nowadays "path integral" usually refers to the Feynman path integral in quantum mechanics.

Remark 1.4. In general, if you are asked to take the integral along a circle or rectangle or something else, unless the problem says otherwise, it's safe to assume that you are supposed to integrate with "positive orientation," that is, in the counterclockwise direction. There are reasons for this convention, but a long screed on it doesn't really belong here. Ask me (or somebody who might know) if you're curious.

Let's work through a simple example.

Example 1. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

what is the integral of f around the circle C_r of radius r traversed counter-clockwise?

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Proof. From our result above, we know that we can use any counter-clockwise parametrization of our circle to find this integral. The easiest one to use is the standard parametrization

$$\gamma(t) = (r\cos(t), r\sin(t)), \quad t \in [0, 2\pi].$$

Note in particular that as t goes from 0 to 2π, we move along the circle counter-clockwise. (Quick quiz: How would you parametrize this if we wanted to integrate clockwise along the circle?)

Now, by the theorem we have

$$\begin{aligned}
\int_{C_r} f \cdot dC &= \int_0^{2\pi} \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right) \Bigg|_{(r\cos(t), r\sin(t))} \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2\cos^2(t) + r^2\sin^2(t)}, \frac{2r\sin(t)}{r^2\cos^2(t) + r^2\sin^2(t)} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2}, \frac{2r\sin(t)}{r^2} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(-\frac{2r^2\cos(t)\sin(t)}{r^2} + \frac{2r^2\sin(t)\cos(t)}{r^2} \right) dt \\
&= \int_0^{2\pi} 0\, dt \\
&= 0.
\end{aligned}$$

□

Example 2. Show that the two paths

$$\begin{aligned}
&\gamma(t) = (t, t), \quad t \in [0, 2] \\
&\alpha(t) = (t(2t-3)^2, t(2t-3)^2), \quad t \in [0, 2]
\end{aligned}$$

trace out the same path with the same orientation.

Proof. We observe that

- $t(2t-3)^2 \Big|_0 = 0$,
- $t(2t-3)^2 \Big|_1 = 1$,
- $t(2t-3)^2$ only other root is at $t = 3/2$, and
- $t(2t-3)^2 \Big|_2 = 2(2 \cdot 2 - 3)^2 = 2$.

Therefore, $t(2t-3)^2$ is a continuous function that's 0 at $t = 0$, 2 at $t = 2$, and $\geq$ zero on the entire interval $[0, 2]$. (This last step is justified by the intermediate value theorem, our observation that f has its only zeros at 0 and $3/2$, and the fact that it takes a positive value on either side of $3/2$.

Hence, the graph of $\alpha(t) = (t(2t-3)^2, t(2t-3)^2)$ on $[0, 2]$ will be the same as $\gamma(t)$! This is because it will hit every point of the form (x, x) for $x \in [0, 2]$. Indeed, it will hit some of these points multiple times! Furthermore, it will only hit points of this form. □

A (surprising) corollary of this result is the integrating along either of these paths is the same, even though integrating along α involves going out from $(0,0)$ to $(1,1)$, then back to $(0,0)$, then back again to $(2,2)$.

Warning 1.5. There is a little detail I am sweeping under the rug here. It is true that integrating along the paths above results in the same line integral, but "tracing out the same curve" should be used with some caution. Everything's okay if you don't "overtrace," but if you do trace over the same line more than once, it is not the "same curve." For instance, a path traversing a circle twice will not generally result in the same integral as traversing a circle once.

Essentially, what's happening here with the second curve is that you're going from $(0,0)$ to $(1,1)$, then going back from $(1,1)$ to $(0,0)$, effectively "erasing" your first attempt at a path, and then going from $(0,0)$ to $(2,2)$.

2. Line Integrals With Respect to Arc Length

We can also integrate a *scalar field* over a curve in a slightly different way. We do this by using the following slightly more general version of a line integral.

Definition 2.1. Given a scalar field $f : \mathbf{R}^n \to \mathbf{R}$ and a continuous path $\gamma : [a,b] \to \mathbf{R}^n$, we can define the **line integral with respect to arc length** of f along γ as

$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot ||\gamma'(t)||\, dt.$$

Just like with the normal line integral, this only depends on the curve drawn by γ and not any particular parametrization. What's going on here is that the arclength fudge factor $||\gamma'(t)||$ forces the integral to go along the curve at a uniform rate, so that it doesn't matter, if you, say, move along the curve at a given rate or at three times that given rate.

Let's see an example of this. Everything is just computational.

Example 3. Integrate the function $f(x,y,z) = x^2y^2 + y^2z^2 + x^2z^2$ over the helix $\gamma(t) = (\cos(t), \sin(t), t)$ where $t \in [0, 2\pi)$.

Proof. We just need to apply our definition above.

$$\begin{aligned}
\int_\gamma f(x,y,z)\cdot d\gamma &= \int_0^{2\pi} f(\cos(t),\sin(t),t)\cdot ||(\cos(t),\sin(t),t)'||\,dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t)+t^2\sin^2(t)+t^2\cos^2(t)).||(-\sin(t),\cos(t),1)||\,dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t)+t^2)\cdot\sqrt{\sin^2(t)+\cos^2(t)+1^2}\,dt \\
&= \int_0^{2\pi} \left(\frac{\sin^2(2t)}{4}+t^2\right)\sqrt{2}\,dt \\
&= \int_0^{2\pi} \left(\frac{1-\cos(4t)}{8}+t^2\right)\sqrt{2}\,dt \\
&= \left(\frac{t}{8}-\frac{\sin(4t)}{32}+\frac{t^3}{3}\right)\sqrt{2}\Big|_0^{2\pi} \\
&= \left(\frac{2\pi}{8}-\frac{0}{32}+\frac{(2\pi)^3}{3}\right)\sqrt{2}-0 \\
&= \frac{2\pi\sqrt{2}}{8}+\frac{8\pi^3\sqrt{2}}{3}.
\end{aligned}$$

□

3. Path-Connected Sets

From the examples given above, line integrals seem like pretty wild things. It seems like there's nothing you can do but just sit down, shut up, and compute until you get an answer. However, it turns out that there are some very powerful theorems that lead to some awesome shortcuts in calculation. But to introduce these results, we first need to familiarize ourselves with the notion of a path-connected set.

Definition 3.1. A set $S \subseteq \mathbf{R}^n$ is called **path-connected** if for any two points $\mathbf{x}$, $\mathbf{y} \in S$, there is a continuous path $\gamma : [a,b] \to \mathbf{R}^n$, *contained entirely within* S, such that $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$.

In class (and in Apostol), this is often shortened to **connected**

Remark 3.2. Technically, there is a small lie here, since the definition of "connected" and "path-connected" are distinct concepts in point-set topology. However, path-connectedness implies connectedness in general, so it's safe, at least for this class, to just call path-connected sets "connected."

Let's do some quick examples.

- An open ball? Yes.
- (The union of) Two parallel lines in the plane? No.
- $\mathbf{R}^n$ itself? Yes.
- (The union of) two circles of radius 1 on the unit sphere? Yes, because they must overlap at some point, and they are themselves path-connected.
- An annulus? Yes.
- (The union of) two open balls? Depends. If they are not disjoint, it is path-connected. If they are disjoint, they are not path-connected.

You get the idea. Most of these things you can see intuitively. However, it's not always easy to determine whether a set is path-connected or not.

Example 4. Here's a tricky one. Is this set

$$S := \left\{ (x, y) \in \mathbf{R}^2 \mid (x, y) \text{ is either of the form } (0, y) \text{ or } \left(x, \sin\left(\frac{1}{x}\right)\right)\right\}$$

path-connected?

The graph looks something like this...[sketch]

Solution. It turns out, this set is NOT path-connected! It certainly "looks" connected, but consider the following argument. Let $(0, y)$ be a point on the y-axis, and let $(x, \sin(1/x))$ be any other part of our set. Suppose that there exists a path $\gamma : [0, 1] \to S$ connecting these two points. Then, in order to do this, it will have to travel along all the curve $(t, \sin(1/t))$ from t to 0. But we know that the function $y = \sin(1/x)$ is not continuous at 0, since it constantly oscillates between ± 1 as it approaches zero! Therefore, any map restricted to this function must also not be continuous if its x-coordinate approaches zero; therefore, γ cannot be continuous, and there is not a path!

Since no such path can exist, this set is not path-connected. □

4. Line Integrals and Gradients

Now we can state an important and powerful result.

Theorem 4.1. *Suppose that $S \subseteq \boldsymbol{R}^n$ is an open and path-connected set. Then, the following conditions are equivalent (that is, if one of the statements hold, then all of the statements hold and if one of these statements doesn't hold, then none of these statements hold), for any function $f : S \to \boldsymbol{R}^n$:*

(a) There is a scalar field $F : S \to \boldsymbol{R}$ such that $\nabla F = f$.
(b) The line integral of f over any path $\gamma : [a, b] \to S$ only depends on the endpoints of γ, that is,

$$\int_\gamma f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a)).$$

(c) The line integral of f over any closed path $\gamma : [a, b] \to S$ (i.e. any path γ with $\gamma(a) = \gamma(b)$, is identically zero.

Since we don't have a rigorous way to talk about "all of the paths" γ in a space S yet, the way we usually apply this theorem is to (1) notice that a given function is a gradient, and then (2) deduce that an other-difficult integral is trivially given by evaluating f on its endpoints, or is zero (because the curve is closed).

Example 5. Recall our example from before. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

what is the integral of f around the circle C_r of radius r traversed counter-clockwise?

Proof. Noting that f is the gradient of the function $F(x, y) = \log(x^2 + y^2)$ and that C_r is a closed curve in an open connected set ($\mathbf{R}^2$ itself), we apply the theorem to see that

$$\int_{C_r} f \cdot dC = 0.$$

□

Now, we don't have any surefire methods yet in this course for finding such gradients; mostly it's just recognizing patterns and making intelligent guesses. Failing that, you can always do it the long, straightforward way to get the answer.

However, what's really cool about the theorem is that this works in general. If we saw the above example, we might have just tried to calculate the answer, instead of using the theorem. But we can apply the theorem to curves that we would not want to integrate by hand.

Example 6. Find the line integral of the vector field

$$f(x, y) = (yz, xz, xy)$$

over the curve

$$\gamma(t) = (1, \cos(t), W(t)), \quad t \in [0, 2\pi]$$

where $W(t)$ is a Weierstrass function, defined as

$$W(t) = \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi t)}{2^n}.$$

What's cool is that $W(t)$ is an example of a function that is everywhere continuous but nowhere differentiable.

Proof. If you want to do that directly, good luck!

For those of us that don't want to integrate an infinite sum of cosines, we can simply note that because $\cos(0) = \cos(101^n \cdot 2\pi \cdot 0) = 1$, we have

$$\gamma(0) = (1, \cos(0), W(0)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1)$$

$$\gamma(2\pi) = (1, \cos(2\pi), W(2\pi)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1),$$

and so this curve is closed. Since $f(x, y, z)$ is the gradient of $F(x, y, z) = xyz$, we conclude that the integral is zero! □