MA 1C RECITATION 05/31/12

1. Integration on Surfaces

To understand the title of this section, we need to understand two things: (1) what "integration" means on a surface and (2) what a "surface" is, mathematically speaking. We'll start by answer the first question, which is less fundamental, but easier to approach intuitively, so whenever I say "surface" for now, just imagine your favorite surface in $\mathbf{R}^3$ that you have seen before, like a sphere or a torus.

Suppose we have a surface $S \subset \mathbf{R}^3$ and some function $f : \mathbf{R}^3 \to \mathbf{R}$. How can we define the integral of f over S?

The reason that we're talking about this stuff now, immediately after the change of variables section, is that change of variables is the right the way to look at our situation.

Here's one way to look at it. Suppose that S is parametrized by some function $\phi : R \to S$, with $R \subset \mathbf{R}^2$. Then one natural way to define the integral of f over S is to say that it is the integral of $f \circ \phi$ over R, where we need to compensate for how ϕ "stretches" the area. Namely, we have the following notion of integral.

Definition 1.1. For a surface $S \subset \mathbf{R}^3$ parametrized by some function $\phi(x, y) : R \to S$ with $R \subset \mathbf{R}^2$ and some function $f : \mathbf{R}^3 \to \mathbf{R}$, we define the **integral of f over S** as

$$\iint_S f\, dS = \iint_R f(\phi(x,y)) \cdot \left\| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right\| \, dx\, dy.$$

Namely, we see that $\left\| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right\|$ accounts for the distortion of space. If you think about what this expression means, it actually makes a lot of sense. At the point (x, y), we shift space by $\frac{\partial \phi}{\partial x}$ along x and by $\frac{\partial \phi}{\partial y}$ along y, so it's distorting the area by the magnitude of the cross-product of those two vectors at that point.

2. First Application: Surface Area of a Sphere

Have you ever wondered how mathematicians came up with all of those annoying formulas that you had to memorize for standardized tests? Probably not. But let's use our new technology to rediscover a well-known formula from calculus.

Example 1. What is the surface area of a sphere S^2 (in $\mathbf{R}^3$)?

Solution. Let's parametrize the sphere with spherical coordinates of radius 1, that is,

$$f(\theta, \phi) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi),$$

so $f([0, 2\pi] \times [0, \pi]) = S^2$. Notice that we don't have a one-to-one correspondence between $[0, 2\pi] \times [0, \pi]$ and S^2, accurate, since we have some overlapping, but it turns out that this is content zero, so it doesn't affect our integral calculation.

Date: May 31, 2012.

Next, we calculate

$$\begin{aligned}\left|\left|\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}\right|\right| &= ||(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \times (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)|| \\ &= ||(-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi)|| \\ &= |\sin\phi|\sqrt{\sin^2\phi\sin^2\theta + \sin^2\phi\cos^2\theta + \cos^2\phi} \\ &= |\sin\phi|.\end{aligned}$$

Thus, we compute

$$\int_0^{\pi}\int_0^{2\pi} |\sin\phi|\, d\theta\, d\phi = 2\pi[-\cos\phi]\Big|_0^{\pi} = 4\pi,$$

which gives us the surface area of the 2-sphere, as desired. □

2.1. **Why does this technique always work for surfaces?** The real question here is "What is a surface"? The reason that this way of defining an integral works for surfaces is because when we are talking about surfaces in $\mathbf{R}^3$, we are actually talking about two-dimensional manifolds in embedded in $\mathbf{R}^3$. I have mentioned manifolds a couple of times beforehand in recitation essentially as a lead into what we're trying to do here. (The amazing thing is that manifolds make our intuition for, say, surfaces, work in arbitrary dimensions, with the exact same formalism!) Don't be scared when you hear the word manifold, even though you're not expected to know the definition of it, it's just a mathematically precise way to describe something that occurs very naturally.

A **2-manifold** (the shorthand for 2-dimensional manifold), or **surface**, is a geometric object that "locally looks like $\mathbf{R}^2$." Inituitively, this means that if you "zoom in" any part of a surface, you cannot tell whether you're on the surface in question or $\mathbf{R}^2$.

Examples: sphere, $\mathbf{R}^2$, torus, klein bottle (try embedding in 4-dimensions!), open set in $\mathbf{R}^2$, open set of a surface, closed set of a surface (technically a manifold *with boundary*).

Non-examples: cone, real line (a 1-manifold, but not a 2-manifold), crossed lines (not a manifold, not even 1-dimensional).

Actually, one of the first major theorems you learn in topology is the classification of "closed" (i.e. compact and boundaryless) 2-manifolds: it says that all 2-manifolds, up to topological equivalence (e.g. via stretching or contracting without sharp corners or folding), are just the sphere, a torus, 2-holed torus, or other n-holed torus. In other words, the only thing preserved topologically for closed 2-manifolds is the number of "holes" or "handles." One cool way to prove this is via an intuitive diagrammatic example. [Show torus example.]

To integrate, on say, cones, which occur naturally in many applications, mathematicians usually use a technique called "stratification" to allow for a well-defined notion of integration, but such things are beyond the scope of this class.

3. Interlude/Warning: Integrating 2-forms

On your homework, you're asked to do some integration where you're integrating along, say $dy \wedge dz$ instead of $dy\, dz$. Note that integration by these two things are NOT the same. This is a very common error. In the first, you are integrating a "2-form" and in the second, you are integrating two "1-forms."

However, don't be scared! To do the homework, all you need to know is to recall that

$$\frac{\partial(X,Y)}{\partial(u,v)} := \det \begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v}. \end{bmatrix}$$

(Important! This is NOT the Jacobian. Do not confuse the two.) Then we define

$$\iint_S P\, dy \wedge dz := \iint_T P(\phi(u,v)) \frac{\partial(Y,Z)}{\partial(u,v)}\, du\, dv.$$

Therefore, if you see an integral with 2-forms like $dx \wedge dy$, all you need to do is break it up into summands and use the above formulation to tell you how to compute the integral.

The reason that these things are mentioned, is that integrating general n-forms is an essential part of integration on manifolds and the way that modern science views integration.

If there is interest, I can tell you more about why these wedges works and the formalism behind them. However, this is probably all you need to do the homework problems.

4. Stokes' Theorem

This is essentially Green's theorem for surfaces. (Or more accurately, Green's theorem is just a kind of Stokes' theorem.)

Theorem 4.1 (Stokes' Theorem)**.** *Suppose that S is a bounded surface with boundary given by a positively oriented (i.e. counterclockwise) curve C and $F : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a continuous differentiable function. Then*

$$\iint_S ((\nabla \times F) \cdot n)\, dS = \int_C F \cdot ds$$

where n denotes the unit normal vector at any point on S.

Remark 4.2. If we have a parametrization ϕ of our surface S, we can explicitly write our normal vector n as

$$n = \frac{\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}}{||\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}||}.$$

We want to use Stokes' theorem in essentially the same situations as Green's theorem. However, it tends to be more useful in one direction than the other.

- Bad curve. Turning integrals over bad curves into an nicer one of curls of functions over surfaces.
- Bad function. Turn integrals of bad functions over some curve into integrals of curves over some region.
- We can also go backwards, but it is generally hard to figure out whether an integrand is of the form $(\nabla \times f) \cdot n$ over a surface. Don't try to do this unless you're really stuck or the problem gives you the function explicitly.

Example 2. If $F(x, y, z) = (-xy^2, x^2y, z)$ and S is the sphere cap $\{(x, y, z) : x^2 + y^2 + z^2 = 25, z \geq 4\}$, find the integral of $(\nabla \times F) \cdot n$ over S.

Solution. We could try and integrate over the surface itself, but it's a pretty tricky computation, and you will almost surely make an error somewhere. However, we see that the boundary is nice, so let's try and apply Stokes' theorem to integrate along the boundary instead! Namely, S has boundary

$$\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 25, z = 4\} = \{(x, y, z) : x^2 + y^2 = 3^2, z = 4\},$$

which we can parametrize in the counterclockwise direction by the curve $\gamma(\theta) = (3\cos(\theta), 3\sin(\theta), 4)$. Therefore, by Stokes' theorem, we have

$$\begin{aligned}
\iint_S (\nabla \times F) \cdot n \, dS &= \int_C F \, dC \\
&= \int_0^{2\pi} (-27\cos(\theta)\sin^2(\theta), 27\cos^2(\theta)\sin(\theta), 4) \cdot (-3\sin(\theta), 3\cos(\theta), 0) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin^3(\theta) + 81\cos^3(\theta)\sin(\theta) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta)(\sin^2(\theta) + \cos^2(\theta)) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta) \\
&= \int_0^{2\pi} \frac{81\sin(2\theta)}{2} \, d\theta \\
&= 0.
\end{aligned}$$

□

Example 3. Find the line integral of $F(x, y, z) = (y^2, x^2, xz)$ around the circle of C radius 1 in the xy-plane, oriented counterclockwise from above.

Solution. We note that $\operatorname{curl}(F) = (0, z, 2y - 2x)$. By Stokes' theorem, our line integral is equal to the integral of $\nabla \times F$ over *any* surface with C as a boundary. (Isn't this strange? But it is true!) Let's make this easy for ourselves and choose the disk in the xy-plane, parametrized by polar coordinates. We then compute

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^1 \int_0^{2\pi} (\nabla \times F)(\phi(r, \theta)) \cdot (\partial r \times \partial \theta) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot ((\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0)) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot (0, 0, r) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} 2r^2(\sin\theta - \cos\theta) \, d\theta \, dr \\
&= \int_0^1 (-2r^2[\sin\theta + \cos\theta]\Big|_0^{2\pi} \, dr \\
&= \int_0^1 0 \, dr \\
&= 0.
\end{aligned}$$

□