MA 1C RECITATION 05/16/13

1. Line Integrals

Definition 1.1. For a function $f : \mathbf{R}^m \to \mathbf{R}^n$ and a continuous path $\gamma : [a, b] \to \mathbf{R}^m$, we define the **line integral of f along** γ to be

$$\int_\gamma f\, d\gamma := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

While the line integral might seem to depend on the path, the following result implies that the integral only depend on the curve traced out by the line itself, and not a particular parametrization.

Theorem 1.2. *Suppose that $\gamma : [a, b] \to \boldsymbol{R}^n$ and $\alpha : [c, d] \to \boldsymbol{R}^n$ are two paths such that (1) α and γ have the same image in $\boldsymbol{R}^n$, (2) $\alpha(a) = \gamma(c)$, and (3) both paths traverse their images with the same orientation (roughly, move in the "same direction"). Then for any function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$, we have*

$$\int_\gamma f \cdot d\gamma = \int_\alpha f \cdot d\alpha,$$

provided that either integral exists.

Remark 1.3. In general, if you are asked to take the integral along a circle or rectangle or something else, unless the problem says otherwise, it's safe to assume that you are supposed to integrate with "positive orientation," that is, in the counter-clockwise direction. There are reasons for this convention, but a long screed on it doesn't really belong here. There's some discussion about it in the book, and Prof. Ni covered orientation and nonorientability of the Möbius strip in lecture, but a real answer takes a little more time. Ask me (or somebody who might know) if you're curious.

Let's work through a simple example.

Example 1. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

what is the integral of f around the circle C_r of radius r traversed counter-clockwise?

Proof. From our result above, we know that we can use any counter-clockwise parametrization of our circle to find this integral. The easiest one to use is the standard parametrization

$$\gamma(t) = (r \cos(t), r \sin(t)), \quad t \in [0, 2\pi].$$

Note in particular that as t goes from 0 to 2π, we move along the circle counter-clockwise. (Quick quiz: How would you parametrize this if we wanted to integrate clockwise along the circle?)

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Now, by the theorem we have

$$\begin{aligned}
\int_{C_r} f \cdot dC &= \int_0^{2\pi} \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right) \Big|_{(r\cos(t), r\sin(t))} \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2\cos^2(t)+r^2\sin^2(t)}, \frac{2r\sin(t)}{r^2\cos^2(t)+r^2\sin^2(t)} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(\frac{2r\cos(t)}{r^2}, \frac{2r\sin(t)}{r^2} \right) \cdot (-r\sin(t), r\cos(t))\, dt \\
&= \int_0^{2\pi} \left(-\frac{2r^2\cos(t)\sin(t)}{r^2} + \frac{2r^2\sin(t)\cos(t)}{r^2} \right) dt \\
&= \int_0^{2\pi} 0\, dt \\
&= 0.
\end{aligned}$$

(Conceptual check: What answer would we get if we integrated clockwise instead?)

□

2. Line Integrals With Respect to Arc Length

We can also integrate a *scalar field* over a curve in a slightly different way. We do this by using the following slightly more general version of a line integral.

Definition 2.1. Given a scalar field $f : \mathbf{R}^n \to \mathbf{R}$ and a continuous path $\gamma : [a, b] \to \mathbf{R}^n$, we can define the **line integral with respect to arc length** of f along γ as

$$\int_\gamma f \cdot d\gamma := \int_a^b f(\gamma(t)) \cdot ||\gamma'(t)||\, dt.$$

Remark 2.2. Marsden and Tromba call *this* a **path integral** and cover it before introducing the line integral, but I think this comes conceptually after the definition of the line integral, so I will cover it here.

Just like with the normal line integral, this only depends on the curve drawn by γ and not any particular parametrization. What's going on here is that the arclength fudge factor $||\gamma'(t)||$ forces the integral to go along the curve at a uniform rate, so that it doesn't matter, if you, say, move along the curve at a given rate or at three times that given rate.

Let's see an example of this. Everything is just computational.

Example 2. Integrate the function $f(x, y, z) = x^2y^2 + y^2z^2 + x^2z^2$ over the helix $\gamma(t) = (\cos(t), \sin(t), t)$ where $t \in [0, 2\pi)$.

Proof. We just need to apply our definition above.

$$\begin{aligned}
\int_\gamma f(x,y,z)\cdot d\gamma &= \int_0^{2\pi} f(\cos(t),\sin(t),t)\cdot ||(\cos(t),\sin(t),t)'||\,dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t)+t^2\sin^2(t)+t^2\cos^2(t)).||(-\sin(t),\cos(t),1)||\,dt \\
&= \int_0^{2\pi} (\cos^2(t)\sin^2(t)+t^2)\cdot\sqrt{\sin^2(t)+\cos^2(t)+1^2}\,dt \\
&= \int_0^{2\pi} \left(\frac{\sin^2(2t)}{4}+t^2\right)\sqrt{2}\,dt \\
&= \int_0^{2\pi} \left(\frac{1-\cos(4t)}{8}+t^2\right)\sqrt{2}\,dt \\
&= \left(\frac{t}{8}-\frac{\sin(4t)}{32}+\frac{t^3}{3}\right)\sqrt{2}\Big|_0^{2\pi} \\
&= \left(\frac{2\pi}{8}-\frac{0}{32}+\frac{(2\pi)^3}{3}\right)\sqrt{2}-0 \\
&= \frac{2\pi\sqrt{2}}{8}+\frac{8\pi^3\sqrt{2}}{3}.
\end{aligned}$$

□

3. Integration on Surfaces

To understand the title of this section, we need to understand two things: (1) what "integration" means on a surface and (2) what a "surface" is, mathematically speaking. We'll start by answer the first question, which is less fundamental, but easier to approach intuitively, so whenever I say "surface" for now, just imagine your favorite surface in $\mathbf{R}^3$ that you have seen before, like a sphere or a torus.

Suppose we have a surface $S \subset \mathbf{R}^3$ and some function $f : \mathbf{R}^3 \to \mathbf{R}$. How can we define the integral of f over S?

The reason that we're talking about this stuff now, immediately after the change of variables section, is that change of variables is the right the way to look at our situation.

Here's one way to look at it. Suppose that S is parametrized by some function $\phi : R \to S$, with $R \subset \mathbf{R}^2$. Then one natural way to define the integral of f over S is to say that it is the integral of $f \circ \phi$ over R, where we need to compensate for how ϕ "stretches" the area. Namely, we have the following notion of integral.

Definition 3.1. For a surface $S \subset \mathbf{R}^3$ parametrized by some function $\phi(x,y)$: $R \to S$ with $R \subset \mathbf{R}^2$ and some function $f : \mathbf{R}^3 \to \mathbf{R}$, we define the **integral of** f **over** S as

$$\iint_S f\,dS = \iint_R f(\phi(x,y))\cdot\left|\left|\frac{\partial\phi}{\partial x}\times\frac{\partial\phi}{\partial y}\right|\right|\,dx\,dy.$$

Namely, we see that $\left|\left|\frac{\partial\phi}{\partial x}\times\frac{\partial\phi}{\partial y}\right|\right|$ accounts for the distortion of space. If you think about what this expression means, it actually makes a lot of sense. At the point (x,y), we shift space by $\frac{\partial\phi}{\partial x}$ along x and by $\frac{\partial\phi}{\partial y}$ along y, so it's distorting the area by the magnitude of the cross-product of those two vectors at that point.

3.1. **Application: Surface Area of a Sphere.** Have you ever wondered how mathematicians came up with all of those annoying formulas that you had to memorize for standardized tests? Probably not. But let's use our new technology to rediscover a well-known formula from calculus.

Example 3. What is the surface area of a sphere S^2 (in $\mathbf{R}^3$)?

Solution. Let's parametrize the sphere with spherical coordinates of radius 1, that is,

$$f(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

so $f([0, 2\pi] \times [0, \pi]) = S^2$. Notice that we don't have a one-to-one correspondence between $[0, 2\pi] \times [0, \pi]$ and S^2, accurate, since we have some overlapping, but it turns out that this is content zero[1], so it doesn't affect our integral calculation.

Next, we calculate

$$\begin{aligned}\left|\left|\frac{\partial f}{\partial\theta} \times \frac{\partial f}{\partial\phi}\right|\right| &= ||(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \times (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)|| \\ &= ||(-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi)|| \\ &= |\sin\phi|\sqrt{\sin^2\phi\sin^2\theta + \sin^2\phi\cos^2\theta + \cos^2\phi} \\ &= |\sin\phi|.\end{aligned}$$

Thus, we compute

$$\int_0^\pi \int_0^{2\pi} |\sin\phi|\, d\theta\, d\phi = 2\pi[-\cos\phi]\Big|_0^\pi = 4\pi,$$

which gives us the surface area of the 2-sphere, as desired. $\square$

3.2. **Why does this technique always work for surfaces?** The real question here is "What is a surface"? The reason that this way of defining an integral works for surfaces is because when we are talking about surfaces in $\mathbf{R}^3$, we are actually talking about two-dimensional manifolds in embedded in $\mathbf{R}^3$. I have mentioned manifolds a couple of times beforehand in recitation essentially as a lead into what we're trying to do here. (The amazing thing is that manifolds make our intuition for, say, surfaces, work in arbitrary dimensions, with the exact same formalism!) Don't be scared when you hear the word manifold, even though you're not expected to know the definition of it, it's just a mathematically precise way to describe something that occurs very naturally.

A **2-manifold** (the shorthand for 2-dimensional manifold), or **surface**, is a geometric object that "locally looks like $\mathbf{R}^2$." Inituitively, this means that if you "zoom in" any part of a surface, you cannot tell whether you're on the surface in question or $\mathbf{R}^2$.

Examples: sphere, $\mathbf{R}^2$, torus, klein bottle (try embedding in 4-dimensions!), open set in $\mathbf{R}^2$, open set of a surface, closed set of a surface (technically a manifold *with boundary*).

Non-examples: cone, real line (a 1-manifold, but not a 2-manifold), crossed lines (not a manifold, not even 1-dimensional).

Actually, one of the first major theorems you learn in topology is the classification of "closed" (i.e. compact and boundaryless) 2-manifolds: it says that all

[1] a technical condition, essentially says that the area is "so small" that the integral of anything on it is 0

2-manifolds, up to topological equivalence (e.g. via stretching or contracting without sharp corners or folding), are just the sphere, a torus, 2-holed torus, or other n-holed torus. In other words, the only thing preserved topologically for closed 2-manifolds is the number of "holes" or "handles." One cool way to prove this is via an intuitive diagrammatic example.

To integrate, on say, cones, which occur naturally in many applications, mathematicians usually use a technique called "stratification" to allow for a well-defined notion of integration, but such things are beyond the scope of this class.

4. Check: Which integral should we use?

Example 4. Suppose that we have to integrate the vector field $F(x, y) = (x^2, y^2)$ over the curve $\gamma(t) = (\sin(3t + \pi/4), \sin(t))$ where $t \in [0, 2\pi)$.

Remark 4.1. This kind of curve is called a *Lissajous curve* and has interesting mathematical properties.

Solution. We're not integrating something over a surface, so we need to take one of the line integrals. Since we're integrating on a vector field and not a scalar field, we should take the line integral. We have

$$\begin{aligned}\int_\gamma F(x,y)\,dC &= \int_0^{2\pi} (F\circ\gamma(t))\cdot\gamma'(t)\,dt\\ &= \int_0^{2\pi} (\sin^2(3t+\pi/4), \sin^2(t))\cdot(3\cos(3t+\pi/4), \cos(t))\,dt\\ &= \int_0^{2\pi} 3\cos(3t+\pi/4)\cdot\sin^2(3t+\pi/4)\,dt + \int_0^{2\pi}\cos(t)\cdot\sin^2(t)\,dt\\ &= \int_{\sqrt{2}/2}^{\sqrt{2}/2} u^2\,du + \int_0^0 v^2\,dv\\ &= 0\end{aligned}$$

where we substituted $u = \sin(3t + \pi/4)$ and $v = \sin(t)$ to evaluate the integral. □

5. Line Integrals and Gradients

You may have noticed that many line integrals often give the result zero. There's a good reason why this happens, and a partial explanation comes from the following important and powerful result.

Theorem 5.1. *Suppose that $S \subseteq \boldsymbol{R}^n$ is an open and path-connected[2] set. Then, the following conditions are equivalent (that is, if one of the statements hold, then all of the statements hold and if one of these statements doesn't hold, then none of these statements hold), for any function $f : S \to \boldsymbol{R}^n$:*

(a) *There is a scalar field $F : S \to \boldsymbol{R}$ such that $\nabla F = f$.*
(b) *The line integral of f over any path $\gamma : [a, b] \to S$ only depends on the endpoints of γ, that is,*

$$\int_\gamma f\cdot d\gamma = f(\gamma(b)) - f(\gamma(a)).$$

[2] means that given any two points in the set, there is a path connecting the two points that also lies in the set

(c) *The line integral of f over any closed path $\gamma : [a, b] \to S$ (i.e. any path γ with $\gamma(a) = \gamma(b)$, is identically zero.*

Since we don't have a rigorous way to talk about "all of the paths" γ in a space S yet, the way we usually apply this theorem is to (1) notice that a given function is a gradient, and then (2) deduce that an other-difficult integral is trivially given by evaluating f on its endpoints, or is zero (because the curve is closed).

Example 5. Recall our example from before. For the function

$$f(x, y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

what is the integral of f around the circle C_r of radius r traversed counter-clockwise?

Proof. Noting that f is the gradient of the function $F(x, y) = \log(x^2 + y^2)$ and that C_r is a closed curve in an open connected set ($\mathbf{R}^2$ itself), we apply the theorem to see that

$$\int_{C_r} f \cdot dC = 0.$$

□

Now, we don't have any surefire methods yet in this course for finding such gradients; mostly it's just recognizing patterns and making intelligent guesses. Failing that, you can always do it the long, straightforward way to get the answer.

However, what's really cool about the theorem is that this works in general. If we saw the above example, we might have just tried to calculate the answer, instead of using the theorem. But we can apply the theorem to curves that we would not want to integrate by hand.

Example 6. Find the line integral of the vector field

$$f(x, y) = (yz, xz, xy)$$

over the curve

$$\gamma(t) = (1, \cos(t), W(t)), \quad t \in [0, 2\pi]$$

where $W(t)$ is a Weierstrass function, defined as

$$W(t) = \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi t)}{2^n}.$$

What's cool is that $W(t)$ is an example of a function that is everywhere continuous but nowhere differentiable.

Proof. If you want to do that directly, good luck!

For those of us that don't want to integrate an infinite sum of cosines, we can simply note that because $\cos(0) = \cos(101^n \cdot 2\pi \cdot 0) = 1$, we have

$$\gamma(0) = (1, \cos(0), W(0)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1)$$

$$\gamma(2\pi) = (1, \cos(2\pi), W(2\pi)) = \left(1, 1, \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = (1, 1, 1),$$

and so this curve is closed. Since $f(x, y, z)$ is the gradient of $F(x, y, z) = xyz$, we conclude that the integral is zero! □