MA 1C RECITATION 05/23/13

1. Prelude

I'll write this formula again:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

where Ω is a compact region, $\partial\Omega$ denotes its boundary taken with positive orientation, ω is a differential form, and d is the exterior derivative.

Remember that all of these integral formulas are essentially special cases of this result.

2. Stokes's Theorem

This is essentially Green's theorem for surfaces. (Or more accurately, Green's theorem is just a kind of Stokes' theorem.)

Theorem 2.1 (Stokes's Theorem)**.** *Suppose that S is a bounded surface with boundary given by a positively oriented (i.e. counterclockwise) curve C and $F : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a continuous differentiable function. Then*

$$\iint_S ((\nabla \times F) \cdot n)\, dS = \int_C F \cdot ds$$

where n denotes the unit normal vector at any point on S.

Remark 2.2. If we have a parametrization $\phi(x, y)$ of our surface S, we can explicitly write our normal vector n as

$$n = \frac{\frac{\partial\phi}{\partial x} \times \frac{\partial\phi}{\partial y}}{||\frac{\partial\phi}{\partial x} \times \frac{\partial\phi}{\partial y}||}.$$

We want to use Stokes' theorem in essentially the same situations as Green's theorem. However, in this case, it tends to be more useful in one direction (turning a line integral into a surface integral) than the other direction.

- Bad curve. Turning integrals over bad curves into an nicer one of curls of functions over surfaces.
- Bad function. Turn integrals of bad functions over some curve into integrals of possible nicer functions over some region.
- We can also go backwards, but it is generally hard to figure out whether an integrand is of the form $(\nabla \times f) \cdot n$ over a surface. Don't try to do this unless you're really stuck or the problem gives you the function f explicitly, as in the following example.

Example 1. If $F(x, y, z) = (-xy^2, x^2y, z)$ and S is the sphere cap $\{(x, y, z) : x^2 + y^2 + z^2 = 25, z \geq 4\}$, find the integral of $(\nabla \times F) \cdot n$ over S.

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Solution. We could try and integrate over the surface itself, but it's a pretty tricky computation, and you will almost surely make an error somewhere. However, we see that the boundary is nice, so let's try and apply Stokes' theorem to integrate along the boundary instead! Namely, S has boundary

$$\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 25, z = 4\} = \{(x, y, z) : x^2 + y^2 = 3^2, z = 4\},$$

which we can parametrize in the counterclockwise direction by the curve $\gamma(\theta) = (3\cos(\theta), 3\sin(\theta), 4)$. Therefore, by Stokes' theorem, we have

$$\begin{aligned}
\iint_S (\nabla \times F) \cdot n \, dS &= \int_C F \, dC \\
&= \int_0^{2\pi} (-27\cos(\theta)\sin^2(\theta), 27\cos^2(\theta)\sin(\theta), 4) \cdot (-3\sin(\theta), 3\cos(\theta), 0) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin^3(\theta) + 81\cos^3(\theta)\sin(\theta) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta)(\sin^2(\theta) + \cos^2(\theta)) \, d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta) \\
&= \int_0^{2\pi} \frac{81\sin(2\theta)}{2} \, d\theta \\
&= 0.
\end{aligned}$$

□

Example 2. Find the line integral of $F(x, y, z) = (y^2, x^2, xz)$ around the circle of C radius 1 in the xy-plane, oriented counterclockwise from above.

Solution. We note that $\operatorname{curl}(F) = (0, z, 2y - 2x)$. By Stokes' theorem, our line integral is equal to the integral of $\nabla \times F$ over *any* surface with C as a boundary. (Isn't this strange? But it is true!) Let's make this easy for ourselves and choose the disk in the xy-plane, parametrized by polar coordinates. We then compute

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^1 \int_0^{2\pi} (\nabla \times F)(\phi(r, \theta)) \cdot (\partial r \times \partial\theta) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot ((\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0)) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot (0, 0, r) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} 2r^2(\sin\theta - \cos\theta) \, d\theta \, dr \\
&= \int_0^1 (-2r^2[\sin\theta + \cos\theta]\Big|_0^{2\pi} \, dr \\
&= \int_0^1 0 \, dr \\
&= 0.
\end{aligned}$$

□

Example 3. Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ x, y, z \geq 0\}$ be an octant of the unit sphere and let $C^+ = \partial S$ be the boundary of S traversed in the counterclockwise direction viewed from the positive z-axis. if $F(x, yz) = (x^4, y^4, z^4)$, what is $\int_C F \cdot dC$?

Solution. One approach here is to just take line integrals. Namely, we can parametrize C as three curves

$$\begin{aligned}\gamma_1(t) &= (\cos(t), \sin(t), 0)\\ \gamma_2(t) &= (0, \cos(t), \sin(t))\\ \gamma_3(t) &= (\sin(t), 0, \cos(t))\end{aligned}$$

where $t \in [0, \pi/2]$. Note that these traverse the curve in the counterclockwise direction. Therefore, we have

$$\begin{aligned}\int_C F \cdot dC &= \sum_{i=1}^{3} \int_0^{\pi/2} (F \circ \gamma_i(t)) \cdot (\gamma'(t))\, dt\\ &= \sum_{i=1}^{3} \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t)\, dt\\ &= 3\int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t)\, dt\\ &= \left[-3\int_0^{\pi/2} \cos^4(t)\sin(t)\, dt\right] + \left[3\int_0^{\pi/2} \sin^4(t)\cos(t)\, dt\right].\end{aligned}$$

To evaluate these last two integrals we u-substitute $u = \cos(t)$ for the first integral and $u = \sin(t)$ for the latter integral, so that we obtain

$$\begin{aligned}\int_C F \cdot dC &= \left[3\int_1^0 u^4\, du\right] + \left[3\int_0^1 u^4\, du\right]\\ &= -3\int_0^1 u^4\, du + 3\int_0^1 u^4\, du\\ &= 0.\end{aligned}$$

Alternatively, we could have used Stokes' theorem, which tells us the integral of F over C is the integral of $(\nabla \times F) \cdot \mathbf{n}$ over S. However, we have

$$\begin{aligned}\mathrm{curl}(F) &= \nabla \times F\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &= (0 - 0, 0 - 0, 0 - 0)\\ &= (0, 0, 0).\end{aligned}$$

Therefore, $(\nabla \times F) \cdot \mathbf{n}$ must be zero and so the integral of it over S must be zero. □

Example 4. Evaluate the surface integral $\iint_S (\nabla \times F) \cdot n\, dS$ where S is the top hemisphere of the unit sphere, and

$$F = (x^2, xy, xz).$$

Solution. By Stokes's theorem, the integral of the curl is the line integral of F around the boundary. Note that the orientation is counterclockwise, when looking at it from above, so we can parameterize it by

$$c(t) = (\cos(t), \sin(t), 0), \quad t \in [0, 2\pi].$$

Therefore, the line integral is

$$\begin{aligned}\int_0^{2\pi} F \cdot ds &= \int_0^{2\pi} (\cos^2(t), \cos(t)\sin(t), 0) \cdot (-\sin(t), \cos(t), 0)\, dt \\ &= \int_0^{2\pi} \cos^2(t)\sin(t) - \cos^2(t)\sin(t)\, dt \\ &= 0.\end{aligned}$$

□