MATH 2A RECITATION 10/04/12

1. Administrivia

I'm Brian, and I'll be your Math 2a TA this quarter. Here's some basic info.

Email: bhwang at caltech
Website: http://hwang.caltech.edu/math2a/
Office: 385 Sloan
Office Hours: Mondays 8-9pm. There are several TAs for the class, be sure to check out other times if you need some help!

Homework is due **Tuesdays at noon**. Please do not be late. You get one free extension of a day, if you email me beforehand by midnight the night before it's due. After that, any late assignments (which must be submitted to your section TA's mailbox) are half-credit until a week afterwards, and no credit after that.

You are allowed (and encouraged!) to use programs like Matlab and Mathematica to do things like draw direction fields. They are available for free to Caltech students from the IMSS website. But do not use a computer to do symbolic manipulation (like symbolically solving differential equations).

Don't worry about writing down everything I say. I will post lecture notes on my site.

My philosophy towards recitations is to focus on the practical skills that you'll need to do the homework. All the beautiful theory and context will be left to Prof. Marcolli in your regular lectures. Here, I'll focus on the essentials, and emphasizes the quick, dirty skills that you will need to succeed in the class.

The style of a differential equations class is slightly different from other math subjects you may have taken: it's not all building up to one big theory, like calculus did with respect to differentiation and integration; instead you learn several techniques, and the game is to figure out which one to use in each situation.

1.1. Your Mindset.

So how should approach differential equations in this class? Differential equations are like Pokémon, little monsters. Scary when you first meet them, but extremely useful, powerful, and even cute once you get familiar with them.

Now, what was the trick to Pokémon? The trick was that every monster had a unique weakness. You want to enter the class with this mindset. The most important skill to learn in this class: **Train yourself to find each differential equation's weakness**, that is, the quickest and most accurate way to solve it. You do this by carefully practicing and understanding the material. It pays to understand why the tricks work.

We will start off easy and gently, with weak monsters and simple solutions. But soon you'll get to the stronger ones, the trickier ones, the ones that seem almost impenetrable, that seem to have no weakness at all.

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Realize that this is perfectly natural. In the real world, you'll often find differential equations you can't solve, ones that nobody knows how to attack. But you're not in the real world. You're in Math 2a. The homework problems in this class are all canned problems with neat solutions. Previous generations of mathematicians have tamed these monsters, and we know a lot about them, but that doesn't mean they're pushovers! But keep in mind, all of these monsters have chinks in their armor. Let's go find some.

2. How Do You Solve Differential Equations?

Solving differential equations is hard. Remember how hard integration was? Integration is just the "trivial" case for differential equations, the most basic case of the theory. Don't make it harder than it needs to be. Find the trick! We have several ways to solve differential equations, in increasing order of sophistication.

(1) Guessing
(2) Better Guessing (a.k.a. Formulas)

Actually, that's about it. Even formulas are just the best guesses by smart people, ones that people found useful enough to pass down from generation to generation.

2.1. **Why is guessing OK?.**

Suppose we have an Initial Value Problem (IVP), and we manage to guess an answer that works. Is that the only answer? The next result justifies our highly scientific method. It is arguably the most important thing you will learn in this class.

Theorem 1. *(Existence and uniqueness of solutions to ODEs)*
Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$ and $\gamma < y < \delta$, then for any $t_0 \in (\alpha, \beta)$ and $y_0 \in (\gamma, \delta)$, the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution on some open interval $I \subset (\alpha, \beta)$ containing t_0.

In other words, if your guess works, you have enough extra info, and the equation is nice enough, then your solution is unique. It turns out that an analogous theorem holds in general.

Now, not all differential equations have solutions, not even the ordinary ones. But if an equation satisfies the hypotheses of the theorem, there must exist a *unique* solution.

Let's see an example of a differential equation that has more than one solution.

Example 2. Consider the IVP

$$y' = y^{1/3}, y(0) = 0.$$

Now, $f(t, y) = y^{1/3}$ is continuous everywhere, but $\partial f/\partial y$ does not exist when $y = 0$. Hence, we do not have continuity of $\partial f/\partial y$, and so the theorem doesn't work.

For instance, what is an obvious solution to the differential equation? $y = 0$. Also, $y = \pm(\frac{2}{3}t)^{3/2}$ are solutions for $t \geq 0$.

However, this theorem has the potential to save us a lot of work. If we've found a solution and the theorem holds, we do not need to waste our time looking for more solutions.

Now, a guess by itself won't qualify for much on the homework or on an exam, but if you find something that works and the hypotheses of the theorem holds, you know you've got your answer. Part of your mathematical education is to learn how to think like a mathematician, and that means learning how to write like one. But if you have a solution that works, it's not hard to work backwards and give a complete answer.

Summary: Check the conditions of your equations. Identify any discontinuities. If you can't find any, you can safely guess by our theorem.

Let's try it on a problem that you're not yet expected to know how to solve.

Example 3. Consider the following IVP:

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Let's take a wild guess. What's something you remember that becomes its negative when you differentiate twice? Let's try $y = \sin t$. We then see that

$$(\sin t)'' + \sin t = -\sin t + \sin t = 0$$

and furthermore, $y(0) = 0$ and $y'(0) = \cos(0) = 1$. Since the hypotheses for our theorem holds, by the existence and uniqueness theorem, this is the indeed the only solution to the problem.

What about $3\cos t$? What about $\cos t$? What about $\cos t + \sin t$? What about e^x? What about something crazy, like $102931\cos t - 124834\sin + \pi^{52}e^x$? All of them almost work. But it doesn't work with the initial values. Hence, we see that a solution to a differential equation depends heavily on the initial conditions, the extra data, that we are given.

You'll find out a systematic way to deal with equations like the one above in the near future. (Next week?) But this gives you an idea of why the theorem holds and why all of the conditions are needed.

3. First-Order ODEs and Integrating Factors

Consider a first-order linear ODE:

$$y' + p(t)y = g(t).$$

What is the integrating factor? Despite all the manipulation seen in the book, the actual answer is marvelously simple:

$$\mu(t) = e^{\int p(t)\,dt}.$$

Piece of cake, right? Why does this work? It's because we're using the product rule as a simplifying mechanism.

3.1. **Monster 1. First-Order ODEs.**

Every time you see one, hit it with an integrating factor. Let's see this in action.

Example 1. Let's solve the differential equation

$$y' - \frac{3}{t}y = 0.$$

The integrating factor is

$$\mu(t) = \exp\left(\int -\frac{3}{t}\,dt\right) = \exp\left(-3\ln t\right) = t^{-3}.$$

Multiplying through, we obtain

$$\frac{y'}{t^3} - \frac{3}{t^4}y = 0.$$

In other words,

$$\left(\frac{y}{t^3}\right)' = 0$$

by the product rule. Integrating, we obtain

$$\frac{y}{x^3} = C,$$

so we have the solution $y(x) = Cx^3$.

4. Separable Equations

If you need to solve a differential equation of the form $\frac{dy}{dt} = f(y)g(t)$, then you can write $\frac{1}{f(y)}dy = g(t)dt$. If you integrate this, you get an (implicit) solution to the differential equation. Observe that I've "multiplied" by dt, which doesn't make any sense. However, writing an equation in this form is justified in the book, as well as in any other book on differential equations. To truly understand what's going on, you'll need to take some more advanced math courses. But for now, remember not to say things like "multiply through by dt." Instead say something like, "we can write the equation in the following form."

Let's see how this works.

Example 1. We want to solve $y' = ty^2$. We can write the ODE in the form

$$\frac{1}{y^2}\,dy = t\,dt$$

so by integrating

$$\int y^{-2}\,dy = \int t\,dt$$

we obtain

$$-\frac{1}{y} = \frac{t^2}{2} + C$$

for some constant C. This gives us an implicit solution. With some algebraic manipulation, we find that

$$y(t) = -\frac{2}{t^2 + D}$$

for some constant D.

Therefore, if you see an ODE that you can "separate" in this way, isolating terms of one variable on one side, and of the other variable on the other side, then you easily integrate them to get your desired solution.

5. Tips on HW

Lastly, here are a couple of tips for the homework, pointing out some common errors.

Start early. Don't make this harder than it needs to be.

Justify all of your steps. For example, if you are dividing by t, make sure that say that you are not in an interval where t can be 0! Also, make sure that you don't just say you found *a* solution. Go the extra step and state why you have found *all* the solutions. See where your answer might break down, and be careful about assumptions.

Write in complete sentences. This helps clarify your own thinking and makes the grading much easier, and we'll pass the good will back to you!