MATH 2A RECITATION 10/18/12

1. Linear Algebra Review

We'll begin by doing a quick review of the relevant linear algebra.

Definition 1. A finite subset of n vectors $v_1, \ldots, v_n$ is *linearly independent* if we have a linear combination

$$a_1 v_1 + \cdots + a_n v_n = 0$$

then that $a_1 = a_2 = \cdots = a_n = 0$. In other words, this is only such way to express 0.

Theorem 2. ***(The Grand Theorem of Linear Algebra)*** *Suppose we are given a system of n linear equations in n variables, and let $\boldsymbol{Ax} = \boldsymbol{b}$ denote the corresponding matrix equation. Then the following are equivalent:*

(1) $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution
(2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution
(3) $\boldsymbol{A}$ can be row-reduced to the identity matrix
(4) the columns of $\boldsymbol{A}$ are linearly independent
(5) the rows of $\boldsymbol{A}$ are linearly independent
(6) the dimension of the column space of $\boldsymbol{A}$ is n
(7) the dimension of the row space of $\boldsymbol{A}$ is n
(8) $\det(\boldsymbol{A}) \neq 0$
(9) the eigenvalues of $\boldsymbol{A}$ are nonzero
(10) $\boldsymbol{A}$ is invertible
(11) the transpose $\boldsymbol{A}^t$ is invertible.

In particular, note that **A** has linearly independent columns if and only if the determinant of **A** is nonzero. Remembering this is the first step to understand the theory of the Wronskian.

2. The Wronskian

The Wronskian is a determinant of a particular matrix and, when it works, provides a useful way to show that a set of functions is linearly independent in a given interval.

But what is the Wronskian? It is a little less clear than I would like it to be in the book, so I have written out a concrete definition.

Definition 1. Given n functions $f_1, \ldots, f_n$, the Wronskian $W(f_1, \ldots, f_n)$ is given by

$$W = W(f_1, \ldots, f_n) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

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So the Wronskian is the just the determinant of a particular matrix. This square matrix is often called the *fundamental matrix* (especially when the determinant is nonzero, for reasons to be made clear later).

Remark 2. There is also a slightly different definition of fundamental matrix in your book, but you still take its determinant to get the Wronskian. It's a good exercise to understand the connection between this fundamental matrix here and the fundamental matrix defined in your book. (Ask me if you can't figure this out.)

Remark 3. Sometimes the fundamental matrix is flipped, with the derivatives lying above the entries in the vector. However, note that this will give us the same answer (up to sign), because exchanging adjacent rows in a matrix just switches the sign of the determinant.

Actually, as far as we're concerned, the Wronskian is only well-defined up to constant. For instance, we could easily replace $f_1(t)$ with $cf_1(t)$ for some nonzero constant c and our techniques will still work. This is why we are only interested in the vanishing behavior of the Wronskian, that is, whether it is zero or nonzero.

But why is this weird determinant useful? It's because it encodes essential information about the linear independence of the solutions.

2.1. **The Wronskian and Linear Independence.** If the functions f_i are linearly dependent, then the columns of the Wronskian are linearly dependent as well, because differentiation is a linear operation. Hence, $W(f_i) = 0$. Thus, we can use the Wronskian to show that a set of differentiable functions is linearly independent on an interval by showing that it does not vanish identically. Let's see an example.

Example 4. Are $\sin t$ and $\cos t$ linearly independent?

This is interesting if you think about it, because $\sin t$ and $\cos t$ are very similar functions, one function is just the other shifted by $\pi/2$. However, it turns out that $\sin t$ and $\cos t$ are linearly independent. One way to see this is to consider the power series expansions. Another way is by computing the Wronskian.

Let $f_1(t) = \cos t$ and $f_2(t) = \sin(t)$. Then $f_1'(t) = -\sin t$ and $f_2'(t) = \cos t$ and so we compute

$$\begin{aligned} W(\cos t, \sin t) &= \det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \\ &= (\cos t)(\cos t) - (-\sin t)(\sin t) \\ &= \sin^2 t + \cos^2 t \\ &= 1. \end{aligned}$$

Since the Wronskian is nonzero, it follows that $\sin t$ and $\cos t$ are linearly independent.

Example 5. Are $f(x) = 9\cos(2x)$ and $g(x) = 2\cos^2(x) - 2\sin^2(x)$ linearly independent?

Consider the equation

$$c(9\cos(2x)) + k(2\cos^2(x) - 2\sin^2(x)) = 0.$$

We need to determine if we can find nonzero constants c and k that will make this true for all x. This is generally hard to do. For this equation, you either saw the trick instantly or you were stuck. You can simplify this process by having a good intuition for the solutions or using any number of standard tricks, but it's hard to do this in general.

In this case, we recall that

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

and use this to show that our equation is

$$(9c + 2k)\cos(2x) = 0.$$

Hence, $c = -2$ and $k = 9$ are one of infinitely many solutions that work, showing that $f(x)$ and $g(x)$ are not linearly independent.

We then expect that the Wronskian must be zero. Indeed, since $f'(x) = -18\sin(2x)$ and $g'(x) = -4\cos(x)\sin(x) - 4\sin(x)\cos(x) = -8\sin(x)\cos(x)$. Therefore,

$$\begin{aligned} W(f,g) &= \det \begin{bmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -8\sin(x)\cos(x) \end{bmatrix} \\ &= -72\cos(2x)\sin(x)\cos(x) + 36\sin(2x)(\cos^2(x) - \sin^2(x)) \\ &= -72\cos(2x)\sin(x)\cos(x) + 36(2\sin(x)\cos(x))(\cos(2x)) \\ &= 0 \end{aligned}$$

as expected.

Hence, we see that linear dependence (at some point) implies that the Wronskian is zero. However, the following remark is very important.

Remark 6. If the Wronskian is *identically zero* over the interval, the functions *may or may not* be linearly independent. You can't say anything here without further reasoning, like finding an explicit solution, as in the example.

A common mistake is assuming that $W \equiv 0$ everywhere implies linear dependence. This is not true. The classical example is to consider x^2 and $x|x|$. They have continuous derivatives and their Wronskian vanishes everywhere, but they are not linearly dependent in a neighborhood of 0.

This remark is important, so take some time to absorb it. It's not emphasized in your text, but the wording of the results hints at their subtlety. Recall the relevant theorems about Wronskians from the book. Let's interpret them very carefully.

Theorem 7. *If $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}$ are solutions of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ on the interval $\alpha < t < \beta$, then in this interval $W[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}]$ either is identically zero or else never vanishes.*

Note the seemingly strange and awkward wording of the conclusion: "either is identically zero *or* never vanishes." This is not a mistake, an oversight, or an indicator of poor writing. It is needed in order for the theorem to be true. It says that if you have a linearly independent set of solutions at a point, the Wronskian can be identically zero *or* nonzero.

Indeed, we often apply the theorem in its its contrapositive form, which is logically equivalent.

Theorem 8. *Suppose that $W[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}]$ is not identically zero and vanishes at some $\alpha < t < \beta$, then $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}$ are not solutions of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$.*

In other words, if the Wronskian is not identically zero, but is, say, a function that is zero at some point in the interval, then the solutions are linearly dependent.

To reiterate, if you just know the Wronskian is zero, you can't say much about linear independence or dependence without more work.

These theorems are powerful because of two reasons: (a) it means that we don't have to check W at every point in the interval, and (b) we can determine whether $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ is a fundamental set of solutions (that is, a linearly independent set of solutions to our system of differential equations) by simply evaluating their Wronskian at any point in the interval.

We'll see the importance of only evaluating a point in the interval, illustrated in the new section.

2.2. **Abel's Identity.** Now, in the special situation that we have, that is, linear differential equations, there's a shortcut to calculate the Wronskian called *Abel's Identity*. This is quite powerful.

When the functions f_i are solutions of a linear differential equation, the Wronskian can be found explicitly by using Abel's identity, even if the functions f_i are not known explicitly.

There's a general formula, but for this class, it's probably best to understand the following simplified case.

Definition 9. Consider a homogeneous linear second-order ODE:

$$y'' + p(t)y' + q(t)y = 0$$

on some interval I with $p(t)$ a continuous function. (We can assume that we are working in either a real or complex situation.) Abel's identity says that if $y_1(t)$ and $y_2(t)$ are two solutions of this differential equation, and given any $t_0 \in I$, the Wronskian can be expressed as

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp\left(-\int_{t_0}^{t} p(u)\, du\right)$$

for all $t \in I$.

Example 10. Consider

$$y'' + y = 0$$

with the interval I being the real line. Then $f_1(x) = \sin(x)$ and $f_2(x) = \cos(x)$ from above are solutions. Recall the Wronskian was identically 1, so this formula holds.

We can see that the same holds for our second example above, because it is identically zero everywhere.

We now state the general form of Abel's identity.

Theorem 11. *Consider a homogeneous linear nth order ODE:*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

on an interval I. Let $y_1, \dots, y_n$ be solutions to our differential equation. Given any $x_0 \in I$, the Wronskian $W(y_1, \dots, y_n)$ then satisfies the relation

$$W(y_1, \dots, y_n)(x) = W(y_1, \dots, y_n)(x_0) \exp\left(-\int_{x_0}^{x} p_{n-1}(u)\, du\right)$$

for all $x \in I$.

Remark 12. In particular, we see that this implies that the Wronskian is either identically zero of it is different from zero at every point $t \in I$. Again, this is why the theorem above is worded the way it is.

If it is different from zero at every point $t \in I$, then the two solutions y_1 and y_2 are linearly independent.