MATH 2A RECITATION 10/25/12

1. Nonhomogeneous Linear ODEs

We are now going to learn how to solve nonhomogeneous linear ODEs. There are a number of techniques to solve these, and we'll go over three that are extremely useful: variation of parameters, undetermined coefficients, and power series methods.

The methods that we will learn this week aren't tricky, but the calculations do get lengthy, so you want to check your answers and be diligent about your bookkeeping. Be careful out there!

2. Variation of Parameters

Theorem 1. *Consider the differential equation*

$$y'' + q(t)y' + r(t)y = g(t).$$

Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for the homogeneous equation

$$y'' + q(t)y' + r(t)y = 0.$$

Then a particular solution for the nonhomogeneous differential equation is

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt.$$

Remark 2. Despite what the solution above looks like, it doesn't matter which one is $y_1(t)$ and which one is $y_2(t)$. Like most techniques in this class, it is independent of choice. You'll end up which the same answer either way.

You'll see why there's a $-y_1$ in the formula above for 2nd order linear ODEs once we introduce the formula for general order.

2.1. How do you use VOP?.

(1) Check that you have a *linear* ODE.
(2) Check that your highest order term has coefficient 1.
(3) Find a fundamental set of solutions to the associated homogeneous equation. (Denote the complimentary solution by y_c.)
(4) Compute the Wronskian. It should be nonzero.
(5) Plug your set of solutions into the formula to get a particular solution y_p.
(6) If needed, add our result from the formula to the solution from the homogeneous case to get the general solution $y = y_c + y_p$.

Let's see an example.

Example 3. Find a general solution to the differential equation:

$$2y'' + 18y = 6\tan(3t).$$

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Solution. We note that we have a linear ODE. Since our formula for variation of parameters requires us to have a coefficient of 1 for the y'' term, we need to divide our equation above by two. Hence, we will be solving the equation

$$y'' + 9y = 3\tan(3t).$$

Suppose we are given the solution for this differential equation:

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t).$$

You can check this this is indeed a solution to our homogeneous equation.

Therefore, we have

$$y_1(t) = \cos(3t) \text{ and } y_2(t) = \sin(3t).$$

Taking the Wronskian, we see that

$$\begin{aligned} W := W(y_1, y_2) &= \det\begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} \\ &= 3\cos^2(3t) + 3\sin^2(3t) = 3. \end{aligned}$$

Therefore, by applying our formula for variation of parameters, we see that the particular solution is

$$\begin{aligned} Y_p(t) &= -\cos(3t)\int \frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t)\int \frac{3\cos(3t)\tan(3t)}{3}\,dt \\ &= -\cos(3t)\int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\ &= -\cos(3t)\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\ &= -\cos(3t)\int [\sec(3t) - \cos(3t)]\,dt + \sin(3t)\int \sin(3t)\,dt \\ &= -\frac{\cos(3t)}{3}(\ln|\sec(3t) + \tan(3t)| - \sin(3t)) + \frac{\sin(3t)}{3}(-\cos(3t)) \\ &= -\frac{\cos(3t)}{3}\ln|\sec(3t) + \tan(3t)|. \end{aligned}$$

Therefore, the general solution is

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln|\sec(3t) + \tan(3t)|.$$

□

Remark 4. Be careful with your bookkeeping for these problems. The formula isn't particularly complicated, but you need to make sure to keep your terms straight. For instance, do not move terms that depend on t from the outside of the integrals to the inside of the integrals.

3. Method of Undetermined Coefficients

For this one, it's easier to see the method in action than to describe it.

Example 1. Determine a particular solution to

$$y'' - 4y' - 12y = 3e^{5t}.$$

Proof. We first want to find the complementary solution to the differential equation, which comes from solving the related homogeneous equation

$$y'' - 4y' - 12y = 0.$$

The characteristic equation is of the form

$$r^2 - 4r - 12 = (r-6)(r+2) = 0$$

and so the roots are $r = -2$ and $r = 6$. Thus, the complementary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}.$$

As a general rule, you want to compute the complementary solution first, for reasons to be made clear as do more of these problems.

Let's begin with calculating the particular solution. Since the inhomogeneous part is an exponential, and we know that they do not appear or disappear with differentiation, we guess that a likely form of the particular solution would be

$$y_p(t) = Ae^{5t}$$

for some constant A. Now we want to plug this into the differential equation and see if we can determine the coefficients:

$$\begin{aligned} 25Ae^{5t} - 4(5Ae^{5t}) - 12(Ae^{5t}) &= 3e^{5t} \\ -7Ae^{5t} &= 3e^{5t}. \end{aligned}$$

Thus, for our guess to be a solution, we need to choose an A that matches, namely,

$$A = -\frac{3}{7}.$$

Therefore, a particular solution to the differential equation is

$$y_p(t) = -\frac{3}{7}e^{5t}.$$

$\square$

So how do you solve these?

1. Solve the complementary equation.
2. Guess the inhomogeneous part. (exponentials if there are exponentials, polynomials if you see polynomials, "sin + cos" if you see sine or cosine, etc.)
3. Plug your particular solution into the differential equation.
4. Match coefficients.

4. Series Solutions

The philosophy here is as follows: splitting up (expanding) a function into infinitely many simple parts is easier than dealing with the function itself.

4.1. **Ordinary/Singular Points.** Consider the differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

Definition 1. We say that $x = x_0$ is an *ordinary point* if both

$$\frac{q(x)}{p(x)} \text{ and } \frac{r(x)}{p(x)}$$

are *analytic at* $x = x_0$. That is, these two quantities have Taylor series around $x = x_0$.

If a point is not ordinary, it is called a *singular point.*

Remark 2. If $p(x), q(x), r(x)$ are polynomials, then checking that the two quotients above are analytic reduces to saying that $p(x_0) \neq 0$. Since we usually deal with the polynomial case in this class, you can readily use this fact.

4.2. **Series Solution Method.** The basic idea to finding a series solution to a differential equation is to assume that there exists a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

and then to try and determine the coefficients a_n. However, we can only do this if the point $x = x_0$ is ordinary. Thus, we see that the classification above is very important.

Let's see some examples of this in action. I'll do one example slowly, making lots of comments along the way.

Example 3. Determine a series solution for the following differential equation about $x_0 = 0$:

$$y'' + y = 0.$$

Note that $p(x) = 1$ in this case, so every point is ordinary. We're looking for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We need to plug this into our DE, so we calculate the derivatives

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \end{aligned}$$

Observe that we can renumber these indices to start at $n = 0$, but this is not a good idea here. Generally speaking, if it turns out things become easier by reindexing to start at $n = 0$, we can fix that later.

Now, plug these into our differential equation, so we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

We now want to combine everything into a single series. To do this, we need to get both series starting at the same index, and make sure that the exponents on the x are the same in both series.

A good way to start is to begin by getting matching the exponents of x. It's generally best to get the exponent to be x^n. The second term is already in this form, and we need to shift the first series down by 2 to get the exponent up to this form. By shifting the first power series, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now notice that while shifting, we also got both series starting from the same index. This won't always be the case, but for now, we offer a thanks to the math gods and accept our blessing. We can then add up the two series, to get

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+a_n]x^n = 0.$$

Now recall our trivial observation, which implies that

$$(n+2)(n+1)a_{n+2}+a_n = 0$$

for all n. This gives us a **recurrence relation**, like the ones we saw for difference equations. In particular, notice that we always want to include the values for n, since we won't always get a power series whose index starts at $n = 0$.

Now, let's determine the values of the a_n's. Note that the recurrence relation has two different a_n's, so we can't just solve for a_n and get a formula that works for all n. However, we can use the recurrence relation to determine all but two of the a_n's.

First solve the recurrence relation for the a_n with the largest subscript. Doing this, we obtain

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

for $n = 0, 1, 2, \ldots$.

What do we do next? Just start calculating.

$$\begin{aligned}
n &= 0, \quad a_2 = \frac{-a_0}{2\cdot 1} \\
n &= 1, \quad a_3 = \frac{-a_1}{3\cdot 2} \\
n &= 2, \quad a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1} \\
n &= 3, \quad a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2} \\
n &= 4, \quad a_6 = \frac{-a_4}{6\cdot 5} = \frac{-a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \\
n &= 5, \quad a_7 = \frac{-a_5}{7\cdot 6} = \frac{-a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}.
\end{aligned}$$

Okay, so we're seeing a pattern here: for $k = 1, 2, \ldots$, we have

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \quad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}.$$

Observe that at each step we plugged back in the previous answer, so that when the subscript was even, we could always write a_n in terms of a_0; and when the subscript was odd, we could always write a_n in terms of a_1. This is not always the case.

Furthermore, notice that the formula we developed only works for $k = 1, 2, 3, \ldots$. However, in this particular case, it also works for a_0. Again, this will not always be the case.

You might be worried that we don't know what a_0 and a_1. However, it turns out that we cannot determine them without more information, like an initial value problem.

Now that we have formulas for the a_n's, let's calculate the solution.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} \\ &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \cdots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^{k+1} a_1}{(2k+1)!} x^{2k+1} + \cdots . \end{aligned}$$

We then want to collect all the terms with the same coefficient and then factor out the coefficient:

$$\begin{aligned} y(x) &= a_0 \left[1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \cdots + \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} + \cdots \right] \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \end{aligned}$$

Let's see another example, where I'll skip the waffling.

Example 4. Find a series solution (in terms of powers of x) to the boundary value problem:

$$y'' + 2t^2 y = 0, \quad y'(0) = 0, y(0) = 2.$$

We know the solution is of the form $y = \sum_{i=0}^{\infty} a_i t^i$. Any solution to the problem must have

$$\sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} t^i + 2 \sum_{i=2}^{\infty} a_{i-2} t^i = 0.$$

Since the coefficients must all cancel out, we must have

$$2a_{i-2} = -(i+2)(i+1) a_{i+2}.$$

Furthermore, we have $a_2 = a_3 = 0$.

From the initial conditions, we know that $a_0 = 2$ and $a_1 = 0$. Therefore, $a_i = 0$ if $a \equiv 0 \pmod 4$, and if $i = 4k$, then we know

$$a_i = \frac{2}{-i(i-1)} a_{i-4} = \frac{2}{-i(i-1)} \cdot \frac{2}{-(i-4)(i-5)} \cdots \frac{2}{-4 \cdot 3} a_0$$

so we have

$$a_{4k} = \frac{(-1)^k 2^{k+1}}{\prod_{j=0}^{k-1} (i-4j)(i-4j-1)}.$$