MATH 2A RECITATION 11/08/12

1. Laplace Transform

The Laplace transform of a function $f(t)$ defined for all reals $t \geq 0$ is the function

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

It's an integral transform, like the more familiar Fourier transform.

Moral. Turn a differential equation into an algebra problem, because algebra is easier than calculus.

The Laplace transform doesn't always give us something neat, but it almost always gives us something simpler to solve. It also allows us to solve IVPs that we cannot solve otherwise.

1.1. When should you use the Laplace transform?

- Homogeneous Linear ODEs? No. Often too complicated and makes things longer. These will usually crack with simpler methods.
- "Simple" Nohomogeneous ODEs? Not really. Usually the Method of Undetermined Coefficients or Variation of Parameters gives us a simpler solution, or at least, one that is no more difficult than of the Laplace transform.
- "Complicated" Nonhomogeneous ODEs? This is where the Laplace transform can come in handy. By "complicated," I mean cases like when the nonhomogeneous part $f(t)$ is not continuous.

Let's consider an example where we need to solve an IVP with a step function, which is not continuous. The fastest way (indeed, the only *general* way that we've learned how to do this in class) is to use the Laplace transform.

It's good to have a a table of various Laplace transforms at hand when you're trying to solve a differential equation via this method. In particular for the next example, remember the following two formulas. Let $F(s) = \mathcal{L}\{f(t)\}$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\mathcal{L}\{u_c(T)f(t-c)\} = e^{-cs}F(s)$$
$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c),$$

where $u_c(t)$ is the *unit step function* (Heaviside function) that is 0 for all $t < c$ and 1 for $t \geq c$.

Example 1. Solve the following IVP:

$$y'' - y' = \cos(2t) + \cos(2t - 12)u_6(t), \quad y(0) = -4, y'(0) = 0.$$

Solution. First rewrite the differential equation so we can identify the function that is being shifted:

$$y'' - y' = \cos(2t) + \cos(2(t-6))u_6(t).$$

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Thus, the function being shifted in $\cos(2t)$. Taking the Laplace transform and plugging in the initial conditions, we obtain

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = \frac{s}{s^2+4} + \frac{se^{-6s}}{s^2+4}$$

and

$$(s^2 - s)Y(s) + 4s - 4 = \frac{s}{s^2+4} + \frac{se^{-6s}}{s^2+4}.$$

We now want to solve for $Y(s)$:

$$\begin{aligned}
(s^2 - s)Y(s) &= \frac{s + se^{-6s}}{s^2+4} - 4s + 4 \\
Y(s) &= \frac{s(1+e^{-6s})}{s(s-1)(s^2+4)} - 4\frac{s-1}{s(s-1)} \\
&= \frac{1+e^{-6s}}{(s-1)(s^2+4)} - \frac{4}{s} \\
Y(s) &= (1 - e^{-6s})F(s) - \frac{4}{s}.
\end{aligned}$$

We now need to decompose $F(s)$ into partial fractions:

$$F(s) = \frac{1}{(s-1)(s^2+4)} = \frac{1}{5}\left(\frac{1}{s-1} - \frac{s+1}{s^2+4}\right).$$

Therefore, we have

$$f(t) = \frac{1}{5}\left(e^t - \cos(2t) - \frac{1}{2}\sin(2t)\right).$$

With this information, we can get the solution to our differential equation. Starting with the transformed equation, we have

$$Y(s) = F(s) + F(s)e^{-6s} - \frac{4}{s}$$

and so

$$y(t) = f(t) + u_6(t)f(t-6) - 4$$

where $f(t)$ is as above. □

2. Convolution of functions

The *convolution* $h = f * g$ of two functions f and g is given by

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Intuitively, it's a sort of "blurring" of functions.

Why bother? It's because it gives us a quick way to calculate the inverse Laplace transform. Namely, we have

$$\mathcal{L}\{f * g\} = F(s)G(s), \quad \mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Example 1. Use a convolution integral to find the inverse Laplace transform of

$$H(s) = \frac{1}{(s^2+a^2)^2}.$$

Solution. To use a convolution integral, we need to write $H(s)$ as a product whose terms are inverse Laplace transforms that we know (or are at least very simple). In our case, we have

$$H(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right).$$

Here, we have

$$F(s) = G(s) = \frac{1}{s^2+a^2}$$

and so

$$f(t) = g(t) = \frac{1}{a}\sin(at).$$

By using the convolution integral $h(t)$, we have

$$\begin{aligned} h(t) &= (f * g)(t) \\ &= \frac{1}{a^2}\int_0^t \sin(at - a\tau)\sin(a\tau)\,d\tau \\ &= \frac{1}{2a^3}(\sin(at) - at\cos(at)). \end{aligned}$$

□

What's neat about convolution is that it allows us to solve IVPs with general forcing functions.

Example 2. Solve the following IVP:

$$4y'' + y = g(t), \quad y(0) = 3, y'(0) = -7.$$

Solution. Before we learned about the Laplace transform and convolutions, we couldn't solve such a general IVP. However, with the techniques that we developed, we will be able to solve this IVP and get a solution in terms of $g(t)$.

Begin by taking the Laplace transform of all the terms, and plug in the initial conditions.

$$\begin{aligned} 4(s^2Y(s) - sy(0) - y'(0)) + Y(s) &= G(s) \\ (4s^2+1)Y(s) - 12s + 28 &= G(s). \end{aligned}$$

Note that all we need to do for the forcing function is to write $G(s)$ for its Laplace transform. We now solve for $Y(s)$:

$$\begin{aligned} (4s^2+1)Y(s) &= G(s) + 12s - 28 \\ Y(s) &= \frac{12s-28}{4(s^2+\frac{1}{4})} + \frac{G(s)}{4(s^2+\frac{1}{4})}. \end{aligned}$$

We factored out a 4 from the denominator to prepare for the inverse transform process. To take inverse transforms, we need to split up the first term and rewrite the second term:

$$\begin{aligned} Y(s) &= \frac{12s-28}{4(s^2+\frac{1}{4})} + \frac{G(s)}{4(s^2+\frac{1}{4})} \\ &= \frac{3s}{s^2+\frac{1}{4}} + \frac{7\frac{2}{2}}{s^2+\frac{1}{4}} + \frac{1}{4}G(s)\frac{\frac{2}{2}}{s^2+\frac{1}{4}}. \end{aligned}$$

We can easily take the inverse of the first two terms, but we need to use a convolution integral for the last term. The two functions that we will use are $g(t)$ and $f(t) =$

$2\sin(\frac{t}{2})$. We can shift either of the two functions in the convolution integral, say $g(t)$. By taking the inverse transform, we obtain

$$y(t) = 3\cos(t/2) + 14\sin(t/2) + \frac{1}{2}\int_0^t \sin(\tau/2)g(t-\tau)\,d\tau.$$

□

Therefore, once we have a $g(t)$, all we need to do is take an integral to get a solution. We can see that convolution is very useful here, because we don't have to re-do the whole process for every forcing function. Instead, we can just solve it once in general and then plug in the different forcing functions. However, if the actual IVP changes, for instance, we change the differential equation itself or the initial values, then we do have to redo the whole thing.

3. Dirac Delta Function

There are many ways to define this, but there are three important properties:

(a) $\delta(t-a) = 0$ for $t \neq a$
(b) $\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)\,dt = 1$ for $\epsilon > 0$.
(c) $\int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)\,dt = f(a)$ for $\epsilon > 0$.

This is not actually a function, but a generalized function, called a distribution. For this class, it essentially means that δ doesn't make sense when it's not inside an integral. Our techniques allow us to solve problems with this as our forcing function. First observe that we have

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$$

Example 1. Solve the following IVP:

$$y'' + 2y' - 15y = 6\delta(t-9), \quad y(0) = -5, y'(0) = 7.$$

Solution. We start by taking the Laplace transform of everything:

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 15Y(s) = 6e^{-9s} \; (s^2 + 2s - 15)Y(s) + 5s + 3 \quad = 6e^{-9s}.$$

Now solve for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{6e^{-9s}}{(s+5)(s-3)} - \frac{5s+3}{(s+5)(s-3)} \\ &= 6e^{-9s}F(s) - G(s). \end{aligned}$$

We then take the partial fractions and compute their inverse transforms

$$\begin{aligned} F(s) &= \frac{1}{(s+5)(s-3)} = \frac{1/8}{s-3} - \frac{1/8}{s+5} \\ f(t) &= \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t} \\ G(s) &= \frac{5s+3}{(s+5)(s-3)} = \frac{9/4}{s-3} + \frac{11/4}{s+5} \\ g(t) &= \frac{9}{4}e^{3t} + \frac{11}{4}e^{-5t} \end{aligned}$$

Therefore, our solution is

$$\begin{aligned} Y(s) &= 6e^{-9s}F(s) - G(s) \\ y(t) &= 6u_9(t)f(t-9) - g(t). \end{aligned}$$

□