MATH 2A RECITATION 11/15/12

1. Many Faces of the Dirac Delta Function

You can define the Dirac delta "function" (remember, it's not really a function, but a generalized function called a *distribution*) in many ways. For instance, one way to obtain it is to set

$$f_\eta(x) = \begin{cases} \frac{1}{\eta}, & -\frac{\eta}{2} \le x \le \frac{\eta}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\delta(x) = \lim_{\eta \to 0} f_\eta(x)$. As $\eta \to 0$, its graph becomes "narrower and taller," but the area below the curve is precisely 1 since it has width η and height $1/\eta$, so

$$\int_{-\infty}^{\infty} f_\eta(x)\,dx = 1.$$

Thus, with this definition, we see that we have

$$\delta(x) = 0 \text{ for } x \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

We can take the limit of many functions to get the Dirac delta function. Some common ones you may see in other subjects include

$$f_\eta(x) = \frac{1}{\eta\sqrt{\pi}} e^{-x^2/\eta^2} \quad (\text{"Gaussian"})$$

which is often used wherever you see probability (because these are normal distributions centered at 0), and

$$f_\eta(x) = \frac{1}{\pi} \cdot \frac{\eta}{x^2+\eta^2} \quad (\text{"Lorentzian"}).$$

Another reason the Dirac delta function is important is that it is the (distributional) derivative of the Heaviside step function

$$\frac{du_c(t)}{dt} = \delta(t-c) = \begin{cases} \infty, & t = c \\ 0, & \text{otherwise.} \end{cases}$$

Graphically, this is pretty clear, since the step function "jumps" at c, so its slope is essentially vertical or infinite.

However, we see this explains why we see step functions in the solutions for nonhomogeneous equations with a term involving a Dirac delta function.

Example 1. Solve the IVP

$$y'' + 3y' - 10y = 4\delta(t-2), \quad y(0) = 2, \quad y'(0) = -3.$$

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We begin by taking the Laplace transform:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) - 10Y(s) &= 4e^{-2s} \\ (s^2 + 3s - 10)Y(s) - 2s - 3 &= 4e^{-2s}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} Y(s) &= \frac{4e^{-2t}}{(s+5)(s-2)} + \frac{2s+3}{(s+5)(s-2)} \\ &= Y_1(s)e^{-2t} + Y_2(s). \end{aligned}$$

Taking partial fraction decompositions, we obtain

$$\begin{aligned} Y_1(s) &= \frac{4}{(s+5)(s-2)} = \frac{4}{7}\frac{1}{s-2} - \frac{4}{7}\frac{1}{s+5} \\ Y_2(s) &= \frac{2s+3}{(s+5)(s-2)} = \frac{1}{s-2} + \frac{1}{s+5}. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$\begin{aligned} y_1(t) &= \frac{4}{7}e^{2t} - \frac{4}{7}e^{-5t} \\ y_2(t) &= e^{2t} + e^{-5t} \end{aligned}$$

so the solution is

$$\begin{aligned} y(t) &= y_1(t-2)u_2(t) + y_2(t) \\ &= u_2(t)\left(\frac{4}{7}e^{2(t-2)} - \frac{4}{7}e^{-5(t-2)}\right) + e^{2t} + e^{-5t} \\ &= u_2(t)\left(\frac{4}{7}e^{2t-4} - \frac{4}{7}e^{-5t+10}\right) + e^{2t} + e^{-5t}. \end{aligned}$$

2. Variation of Parameters for $n = 3$.

This is a good time in the course to consolidate the methods that we know so far. You've gotten good practice with the power series method for solving nonhomogeneous equations from the last couple of homeworks. Now it's time to master how to use the other methods we know for solving nonhomogeneous linear equations.

We've seen variation of parameters for $n = 2$ explicitly, and I've said that there is a general version of variation that works for all orders. However, let's specifically discuss how to use variation of parameters to solve for 3rd order non-homogeneous linear differential equations.

Theorem 1. *Consider a non-homogeneous linear differential equation of order 3:*

$$y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$$

Let $y_1(t), y_2(t), y_3(t)$ be a fundamental set of solutions of the corresponding homogeneous equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0.$$

Then a particular solution to our equation is

$$y_p(t) = \sum_{m=1}^{3} y_m(t) \int \frac{W_m(t)g(t)}{W(t)}$$

where W is the Wronskian and W_m is the determinant of the matrix used to calculate the matrix, but by replacing the mth column by $(0,0,1)^t$.

Remark 2. We can similarly define a formula for variation of parameters for non-homogeneous linear ODEs of any order n.

Let's work through an example in full. I will calculate $W_m(t)$ by replacing the mth column with $(0,0,1)^t \cdot g(t)$, because this tends to simplify computations in the example. You can feel free to do this as well, but make sure that in the whole process, you *only multiply by $g(t)$ once.*

Example 3. Solve

$$y''' - 4y' = 12e^{-2t}$$

using variation of parameters.

We first solve the homogeneous equation.

$$r^3 - 4r = r(r^2 - 4) = r(r+2)(r-2) = 0$$

Therefore, $r = 0, \pm 2$ and so

$$y_c(t) = c_1 + c_2 e^{-2t} + c_3 e^{2t}.$$

Now let's find the particular solution. We have

$$y_1 = 1, y_2 = e^{-2t}, y_3 = e^{2t}.$$

We compute the Wronskian

$$\begin{aligned} W(y_1, y_2, y_3) &= \det \begin{bmatrix} 1 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 0 & 4e^{-2t} & 4e^{2t} \end{bmatrix} \\ &= 1 \det \begin{bmatrix} -2e^{-2t} & 2e^{2t} \\ 4e^{-2t} & 4e^{2t} \end{bmatrix} = -8 - 8 = -16. \end{aligned}$$

Then we compute the associated $W_i(t)$'s:

$$\begin{aligned}
W_1 &= \det \begin{bmatrix} 0 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 12e^{-2t} & 4e^{-2t} & 4e^{2t} \end{bmatrix} \\
&= 12e^{-2t} \det \begin{bmatrix} e^{-2t} & e^{2t} \\ -2e^{-2t} & 2e^{2t} \end{bmatrix} \\
&= 12e^{-2t}(2+2) \\
&= 48e^{-2t} \\
W_2 &= \det \begin{bmatrix} 1 & 0 & e^{2t} \\ 0 & 0 & 2e^{2t} \\ 0 & 12e^{-2t} & 4e^{2t} \end{bmatrix} \\
&= 1 \det \begin{bmatrix} 0 & 2e^{2t} \\ 12e^{-2t} & 4e^{2t} \end{bmatrix} \\
&= -24 \\
W_3 &= \det \begin{bmatrix} 1 & e^{-2t} & 0 \\ 0 & -2e^{-2t} & 0 \\ 0 & 4e^{-2t} & 12e^{-2t} \end{bmatrix} \\
&= 1 \det \begin{bmatrix} -2e^{-2t} & 0 \\ 4e^{-2t} & 12e^{-2t} \end{bmatrix} \\
&= -24e^{-4t}.
\end{aligned}$$

We then calculate the coefficients of our particular solution:

$$\begin{aligned}
u_1 &= \int \frac{W_1}{W}\,dt = \int \frac{48e^{-2t}}{-16}\,dt = -3\int e^{-2t}\,dt = -3\cdot\frac{e^{-2t}}{-2} = \frac{3}{2}e^{-2t} \\
u_2 &= \int \frac{W_2}{W}\,dt = \int \frac{-24}{-16}\,dt = \frac{3}{2}t \\
u_3 &= \int \frac{W_3}{W}\,dt = \int \frac{-24e^{-4t}}{-16}\,dt = \frac{3}{2}\int e^{-4t}\,dt = \frac{3}{2}\cdot\frac{e^{-4t}}{-4} = -\frac{3}{8}e^{-4t}.
\end{aligned}$$

Therefore, our particular solution is

$$\begin{aligned}
y_p &= u_1y_2 + u_2y_2 + u_3y_3 \\
&= \left(\frac{3e^{-2t}}{2}\right)(1) + \left(\frac{3t}{2}\right)(e^{-2t}) + \left(\frac{-3e^{-4t}}{8}\right)e^{2t} \\
&= \frac{3}{2}e^{-2t} + \frac{3}{2}te^{-2t} - \frac{3}{8}e^{-2t} \\
&= \frac{9}{8}e^{-2t} + \frac{3}{2}te^{-2t}.
\end{aligned}$$

Now to find our general solution:

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 + c_2e^{-2t} + c_3e^{-2t} + \frac{9}{8}e^{-2t} + \frac{3}{2}te^{-2t} \\
&= c_1 + c_2e^{-2t} + c_3e^{2t} + \frac{3}{2}te^{-2t}
\end{aligned}$$

because we absorbed the $\frac{9}{8}$ constant into c_2.

3. Variation of Parameters vs. Undetermined Coefficients

Example 1. Find solutions to

$$y'' - y = e^t.$$

We will solve this by undetermined coefficients and also by variation of parameters.

We begin by finding the homogeneous solution. The characteristic equation for $y'' - y = 0$ is

$$r^2 - 1 = (r+1)(r-1) = 0,$$

so we have roots $r = \pm 1$. Therefore, the homogeneous solution is

$$y_h = c_1 e^t + c_2 e^{-t}.$$

Solution: (Undetermined Coefficients). The right hand side of our differential equation is e^t and is given by a linear combination independent function e^t. Hence, we guess that the solution is of the form $y = d_1 e^t$.

However, note that the homogeneous solution contains the term $c_1 e^t$. Thus, we should instead guess

$$y = d_1 t e^t.$$

Substitution into $y'' - y = e^t$ gives us

$$2d_1 e^t + d_1 t e^t - d_1 t e^t = e^t.$$

By canceling e^t and equating coefficients of powers of t, we find that $d_1 = \frac{1}{2}$. Therefore, a particular solution is

$$y_p(t) = \frac{te^t}{2}.$$

□

Solution: (Variation of Parameters). Given our homogeneous solution above, we see that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ is a suitable independent pair of solutions.

We calculate the Wronskian

$$W = \det \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^t \end{bmatrix} = -1 - 1 = -2.$$

By applying the variation of parameters formula, we see that

$$\begin{aligned} y_p(t) &= e^t \int \frac{-e^{-t}}{-2} e^t \, dt + e^{-t} \int \frac{e^t}{-2} e^t \, dt \\ &= e^t \int \frac{dt}{2} - e^{-t} \int \frac{e^{2t}}{2} \, dt \\ &= \frac{te^t}{2} - e^{-t} \frac{e^{2t}}{4} + C \\ &= \frac{te^t}{2} - \frac{e^t}{4} + C. \end{aligned}$$

Setting all constants of integration to zero, we have $y_p(t) = \frac{te^t}{2} - \frac{e^t}{4}$. □

Observe that these two methods give two different particular solutions, $y_p(t) = te^t/2 - e^t/4$ and $y_p(t) = te^t/2$ respectively. The solutions differ by the homogeneous solution $-te^t/4$. However, in both cases, the general solution is

$$y = c_1e^t + c_2e^{-t} + \frac{te^t}{2}$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1 and c_2.