MATH 2A RECITATION 11/15/12

1. Announcements

Next week, in lieu of the usual recitation, I will post up some practice problems and solutions that represent things that I think you should be able to do. They won't necessarily reflect anything on the exam itself. If you have any trouble with solving them, feel free to contact me, as usual.

This homework is the last one, and the final exam will be passed out late next week, and will be due December 12 at noon sharp. We will be grading that afternoon, so please do not be late.

There will be a review session next Thursday evening, probably at 7pm or 8pm. There will be an official announcement soon, but if you have a conflict make sure to contact me ASAP.

2. Nonlinear Systems of Equations

We'll start with some motivation for developing more theory to study systems of equations.

The linear ODE for damped oscillations is given as a second order equation of the form

$$x'' + bx' + \omega^2 x.$$

Recall from class last week that we can write this as a system of first order ODEs describing a pendulum:

$$\begin{aligned} x' &= y \\ y' &= -by - \omega^2 x. \end{aligned}$$

In particular, this has only one equilibrium solution: $x = 0, y = 0$.

Now consider the following modification: a damped *nonlinear* pendulum, given by the system

$$\begin{aligned} x' &= y \\ y' &= -by - \omega^2 \sin x. \end{aligned}$$

We still have $x = 0, y = 0$ as an equilibrium solution, but there are actually an infinite number of equilibria, given by $y = 0$ and $x = n\pi$ for $n \in \mathbf{Z}$.

As in lecture last week, we want to determine the stability of these equilibrium solutions, much like how we did stability analysis for single equations. However, since we have some nonlinearity in the second equation, our old methods won't work, so we need some new techniques to analyze the stability of equilibrium solutions.

Here's the setup. Consider the n-dimensional system

$$\mathbf{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^n$$

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where $f : \mathbf{R}^n \to \mathbf{R}^n$. Note that f is not necessarily linear, so we can't just represent it as an $n \times n$ matrix A anymore. Just like in the linear case, we define equilibrium solutions, or *fixed points*, of the system as points $\mathbf{x}_0$ such that $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.

We want to do a stability analysis of the equilibrium points, that is, we want to see what happens to solutions of our system with initial conditions near a fixed point. We can represent a point near a fixed point in the form

$$\mathbf{x} = \mathbf{x}_0 + \xi$$

where the length of ξ gives an indication of how close we are to the fixed point. Generally, we want this ξ to be small, for the sake of accuracy. Say, for illustration, that $|\xi| << 1$.

As the system evolves, ξ will change. The change of ξ in time is in turn governed by a system of equations. We can approximate this evolution as follows. We note that

$$\mathbf{x}' = \xi'.$$

Next, we have

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0 + \xi).$$

We can expand the right side about the fixed point using the multidimensional version of Taylor's theorem that you may recall from Ma 1c. Thus, we have

$$\mathbf{f}(\mathbf{x}_0 + \xi) = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)\xi + O(|\xi|^2),$$

where $D\mathbf{f}$ is the Jacobian matrix and $O(|\xi|^2)$ accounts for all the higher order ξ-terms.

Since $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$, the system of equations becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}_0)\xi.$$

This is called the **linearization** of the system. We can use this to analyze the stability of equilibrium points, since we can understand linear things pretty well and they don't tend to exhibit any strange pathological behavior.

Example 1. Consider the system

$$x' = -2x - 3xy$$
$$y' = 3y - y^2$$

We first determine the equilibrium points:

$$0 = -2x - 3xy = -x(2 + 3y)$$
$$0 = 3y - y^2 = y(3 - y)$$

From the second equation, we must have $y = 0$ or $y = 3$. The first equation then gives $x = 0$ for either case, so we have two equilibrium points:

$$(0, 0) \text{ and } (0, 3).$$

We want to linearize about each fixed point separately. We first compute the Jacobian matrix:

$$Df(x, y) = \begin{bmatrix} -2 - 3y & -3x \\ 0 & 3 - 2y \end{bmatrix}.$$

Case (0,0): We have

$$Df(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

so the linearized equation is

$$\xi' = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \xi.$$

This can be written as the system

$$\xi_1' = -2\xi_1$$
$$\xi_2' = 3\xi_2.$$

This gives us the linearized system about the origin.

Note the similarity with the original system. The linearized equations are constant coefficient equations, and we can use other methods to determine the nature of the equilibrium point. The eigenvalues of the system of $\lambda = -2, 3$. Thus, the origin is a saddle point.

Case (0,3): We proceed as before. We have the Jacobian matrix

$$Df(0,3) = \begin{bmatrix} -11 & 0 \\ 0 & -3 \end{bmatrix}.$$

Here, the eigenvalues are $\lambda = -11, -3$. Thus, this fixed point is a stable node.

3. Lotka-Volterra (predator/prey) equations

There are many models for population dynamics, a few of which are presented in our textbook. There are two standard models: predator-prey (a.k.a. Lotka-Volterra) models and competing species (a.k.a. two species) models.

In the predator-prey model, we usually have one species (the predator) feeding on the other (the prey). Traditionally, the predators are foxes and the prey are rabbits. It takes the form

$$x' = ax - bxy$$
$$y' = -dy + cxy.$$

In this case, we can think of x as the population of rabbits (prey) and y is the population of foxes (predator). Choosing all constants a, b, c, d to be positive, we can describe the terms as follows:

- ax: When left alone, the prey population will grow. So a represents the natural growth rate of the prey.
- $-dy$: When there are no rabbits, the fox population should decay, so this coefficient should be negative.
- $-bxy$: we add a nonlinear term corresponding to the depletion of the prey when the predators are around.
- cxy: the more prey there are, the more food for the predators, so we add a nonlinear term that ives rise to an increase in the fox population.

The competing species model looks similar, except that there are some obvious sign changes, since in that model one species is not feeding on the other.

Let's do a stability analysis of this nonlinear system of ODEs.

We begin, as usual, by finding the equilibrium solutions

$$x(a - by) = 0$$
$$y(-d + cx) = 0.$$

Thus, the origin and $(\frac{d}{c}, \frac{a}{b})$ are fixed points.

We want to determine the stability by linearization around the fixed points. We have two options: we can either expand the right-hand side of each equation in our system, or we can use the Jacobian matrix. We will choose the latter:

$$Df(x,y) = \begin{bmatrix} a - by & -bx \\ cy & -d + cx \end{bmatrix}.$$

By evaluating at each fixed point, we obtain

$$Df(0,0) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$$
$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}.$$

The eigenvalues of the Jacobian at $(0,0)$ are $\lambda = a, -d$, so the origin is a saddle point.

The eigenvalues of the Jacobian at $(\frac{d}{c}, \frac{a}{b})$ satisfy $\lambda^2 + ad$. Thus, this equilibrium point is a center.

Alternatively, we could linearize by expanding the equations about the fixed points. Although this is equivalent to computing the Jacobian matrix, it may sometimes be faster.

Remark 1. All models are wrong, and this is no exception. What are the major problems? For one, neither equilibrium point is stable, and we see by sketching the flow lines that the populations will cycle endlessly. While this behavior has been observed in nature, it is not a common occurrence.

Population dynamics is a fascinating field, so if you want to learn more or just see some awesome applications and further development of some of the concepts from this class, Steven Strogatz's *Nonlinear Dynamics and Chaos* is an awesome introduction to the subject (as well as many other related topics). With the background from Ma 1 and this class, you should be able to dive into this book.

4. Back to the nonlinear pendulum

OK. Now, we can come back and finally tackle our nonlinear pendulum problem.

Example 1. The system was

$$x' = y$$
$$y' = -by - \omega^2 \sin x.$$

There are an infinite number of equilibrium solutions $(n\pi, 0)$ for $n \in \mathbf{Z}$.

The Jacobian matrix is

$$Df(x,y) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{bmatrix}.$$

Evaluating this at the equilibrium points, we have

$$Df(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ \omega^2(-1)^{n+1} & -b \end{bmatrix}.$$

There are two cases to consider: n even and n odd. For the former case, we find the characteristic equation

$$\lambda^2 + b\lambda + \omega^2 = 0.$$

This has roots

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even n values. If there is no damping, then we obtain centers.

Now suppose that n is odd. Then we have

$$\lambda^2 + b\lambda - \omega^2 = 0$$

with roots

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have saddles (or hyperbolic points).

Draw the phase plane for the undamped nonlinear pendulum! Note that we have a mixture of centers and saddles.