MATH 2A RECITATION 12/06/12

Here are some practice problems for the final exam. The solutions are given at the end. They are not necessarily representative of problems on the midterm (a couple are a little too tricky), but instead will try to unify some loose threads in the course and get you to think a little differently about familiar techniques and mathematical objects.

In doing practice problems, you should of course aim for correct solutions, but you should also pay attention to *speed* and do "sanity checks" (does this calculation/answer make sense?) especially if you are doing an involved computation. Many of the book problems have solutions in the back to facilitate this kind of practice.

Always think: *What is the fastest and most accurate way to solve the equation?* Knowing how to do this "on sight" is probably the most valuable thing you can take away from the course. If you see any of these questions and don't immediately know how to solve it, you should consider seriously reviewing the relevant section until you develop this "sense."

1. Problems

Example 1. (a) Find a general solution for

$$x^{(4)} = x.$$

(b) Find a general solution to the system of ODEs:

$$x'(t) = y(t)$$
$$y'(t) = x(t).$$

Remark 1. Things to think about while doing this: How are the system and the first equation related? What does the phase diagram of the system look like?

Example 2. Find a general solution to

$$y^{(4)} + 4y'' = \sin 2t + te^t + 4.$$

Remark 2. Things to think about: what is the fastest way to solve this equation? What is the second-fastest way to do it, so that you have a way to check your solutions if you have extra time?

On that note, what is a *bad* way to try and solve this equation, i.e. is a valid method, but would take much longer to do? What signs can you look for that would indicate that a particular method might not be the most effective one to use?

Example 3. Consider the difference equation given by

$$x_{n+1} = \sin\left(\frac{\pi}{2}x_n\right)$$

where $x_0 \in (0, 1)$. What is $\lim_{n\to\infty} x_n$?

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Remark 3. Things to think about: What is the difference between a difference equation and a differential equation? Is there anything that can happen with one that can't happen with another? Do we do anything different for stability analysis between the two kinds of equations?

Bonus: The set of difference equations and differential equations are both (uncountably) infinite, but is there any way in which one set is "larger" than the other? For instance, given a differential equations, how many difference equations can you "derive" from it? Similarly, given a difference equation, in how many ways can you "complete" it to a differential equation?

Example 4. Consider the differential equation

$$x'' + f(t)x = 0.$$

For which $f(t)$ is the equilibrium position $x = x' = 0$ asymptotically stable?

Remark 4. Things to think about: when is it productive to decompose a second-order differential equation into a system of two first-order differential equations? (I think this is an very practical technique that wasn't particularly emphasized in the homeworks.)

Example 5. Solve the equation

$$x'' + x = f(t).$$

2. Solutions

1. $x = Ae^t + Be^{-t} + C\cos t + D\sin t$; Substitute $\overline{v} = [x, y]^t = e^{\lambda t}\overline{\xi}$. Then the system has a nontrivial solution if and only if $\lambda^4 = 1$. Thus, every solution of the system is of the form

$$x = A_1e^t + B_1e^{-t} + C_1\cos t + D_1\sin t$$
$$y = A_2e^t + B_2e^{-t} + C_2\cos t + D_2\sin t$$

and by plugging it into the system, we see that $A_1 = A_2$, $B_1 = B_2$, $C_1 = -C_2$, and $D_1 = -D_2$.

2. $y = At\sin 2t + Bt\cos 2t + Cte^t + De^t + Et^2$.

3. The function $x = \sin(\frac{\pi}{2}x)$ has has solutions at 0 and 1, so on (0,1), $\sin(\frac{\pi}{2}x) \geq x$. Thus, for $x_0 \in (0, 1)$, the sequence $\{x_n\}$ is monotone. Furthermore, it is bounded above because $\sin(x) \leq 1$ for real x. Since the sequence $\{x_n\}$ is monotone and bounded, it has a limit L. We have $\sin^{-1}(L) = \frac{\pi}{2}L$, that is, L must be fixed point, that is L is 0 or 1. Since the sequence is increasing and we start in $(0, 1)$, the limit must be 1.

4. Actually, the equilibrium position $x = x' = 0$ cannot be asymptotically stable for *any* choice of $f(t)$! One way to argue this is as follows. Suppose we have a fundamental system of solutions around $x = x' = 0$ spanned by $\xi(t)$ and $\eta(t)$. Since it is asymptotically stable, we have $\xi(t) \to 0$ and $\eta(t) \to 0$ as $t \to \infty$, and so the corresponding Wronskian $W(t) \to 0$ as $t \to \infty$. However, the given system is equivalent to the linear system

$$x_1' = x_2$$
$$x_2' = -f(t)x_1$$

with corresponding matrix

$$A = \begin{bmatrix} 0 & 1 \\ -f & 0 \end{bmatrix}.$$

Recall that the Wronskian of a fundamental system of solutions to a homogeneous linear system satisfies the differential equation

$$W' = (\text{Tr } A(t))W.$$

Since the trace of A is 0, the Wronskian $W(t)$ must be a constant, and so we cannot have $W \to 0$ as $t \to \infty$.

5. As usual, there are many ways to solve this problem, but here's my favorite way. Form the corresponding homogeneous system

$$x_1' = x_2$$
$$x_2' = -x_1.$$

It has a system of fundamental solutions given by $x_1 = \cos t, x_2 = -\sin t$ and $x_1 = \sin t, x_2 = \cos t$. Thus, we look for a solution of the form

$$x_1 = c_1(t)\cos t + c_2(t)\sin t$$
$$x_2 = -c_1(t)\sin t + c_2(t)\cos t.$$

To determine c_1 and c_2, we note that we have

$$c_1'\cos t + c_2'\sin t = 0, \quad -c_1'\sin t + c_2'\cos t = f(t)$$

and so

$$c_1' = -f(t)\sin t, \quad c_2' = f(t)\cos t.$$

Applying variation of parameters, we get a solution

$$x(t) = \left[x(0) - \int_0^t f(\tau)\sin\tau\, d\tau\right]\cos t + \left[x_() + \int_0^t f(\tau)\cos\tau\, d\tau\right]\sin t.$$