MATH 2A RECITATION 10/06/11

1. Introduction

Welcome to the next session of Brian's practical guide to differential equations. I'm going to start off with a couple of brief remarks.

From looking at your homeworks and seeing the problems on the one due next week, I recommend that you learn how to use software like Mathematica or Matlab to graph differential equations. Once again, I remind you that you can't use them to solve the equations symbolically for you. However, you should learn how to graph differential equations on the computer. You don't necessarily need to do it to do well in this class, but most of you are majoring in fields where you will see differential equations on a regular basis, so if you learn it now, you won't have to learn it later.

Unfortunately, we've already come to the time where you're expected to use more material on the homework than can be reasonably covered in a single recitation. While I won't be able to cover everything, I want to go over the "greatest hits," the most important or trickiest techniques for the next assignment.

1.1. **Remarks on Week 2.** On the first problem on the second homework set, we're going to study the importance of initial conditions in determining the solution to a differential equation. Remember that example of

$$y'' + y = 0$$

from last lecture? We saw that it depended heavily on initial conditions. You'll get some practice doing this, as well as classifying their dependence on the initial condition e.g. stable, semistable, unstable, asymptotically stable, etc. A good way to learn this is to study the examples from the book very carefully as you do the problem, or even better, look at how it changes once you input it into a computer.

If you notice, you're assigned to read about existence and uniqueness of solutions to ODEs. Since I covered it and the ideas surrounding it in the last lecture, I will not go in-depth into it, trusting that you got the gist of it and will see how the existence/uniqueness theorems change throughout the book.

Finally, population dynamics is another topic that I know Prof. Ramakrishnan will try to cover, but may skip due to lack of time. It is a beautiful and fascinating topic, but I will relegate it to homework, because it's easy to study beautiful things.

Instead, I will focus on the toughest ideas that will form the bulk of your homework this week.

Date: October 6, 2011.

2. Exact Equations

Let's step up the difficulty a little bit. Today we'll talk about **exact** equations. These are just generalizations of separable equations. Namely, an **exact differential equation** is of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

that satisfy the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The easy to way remember what variable to differentiate by is to follow the maxim "don't offend their friend." In other words, if a term is so cozy with x that it carries a small piece of it around with them (dx), you have to differentiate with respect to other variable.

Example 1. Why are exact equations generalizing separable equations? One reason is because separable equations are exact. If M is a function that only depends on x and N is a function that only depends on y, then

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact.

Consider the equation

$$x^3\,dx + 2y\,dy = 0.$$

Then

$$\frac{\partial}{\partial y}x^3 = 0 = \frac{\partial}{\partial x}2y$$

so our equation is exact. ◇

In applications, you need to careful of boundary conditions (as always), making sure that M, N, M_y and N_y are continuous before you can apply the theorem for existence of solutions to exact equations, just like the case for existence and/or uniqueness that we covered in the last class.

2.1. How do you solve exact equations?

Step 1. Check for exactness. (The reason this works is that if your differential equation is exact, then you will have a solution F with $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial x}$ such that

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}.)$$

Step 2. Integrate with respect to one variable, say, x. You will get a solution that has a "constant term" $O(y)$ a function that depends solely on y. (The "constant term" is usually not constant, it's really a function that disappears when differentiating by x, but I'm adopting the terminology from single-variable integration.)

Step 3. Differentiate with respect to the other variable, say, y and compare with the $\frac{\partial F}{\partial y}$ term. You should then be able to determine $O'(y)$ and thus $O(y)$.

We will see this process in action later.

3. Integrating Factors

Now we are getting to some slightly trickier monsters. The way to solve these equations is to use integrating factors, like we did for first-order ODEs, but this time we want to transform it into an exact equation.

This time you can't just use a single tried and true technique like integrating factors for first-order ODEs. However, there is an algorithm and once you get the hang of it, it's not too hard. **This is important**, so remember how to follow these steps on the homework, and train for speed in addition to accuracy. A good way to do this is to read these steps, then put the paper away, get out a blank sheet, and then try and write the steps from scratch, only looking at the very next step. You don't want to keep forgetting and re-consulting the steps, because that costs valuable time. The other, more obvious, way to train is to do a lot of practice problems using the following method.

You need to be able to do things like this almost automatically to do well on the exams. After all, what use is a tool if it's not ready?

The Algorithm.

Step 1. Check for exactness. In other words, compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ and see if they match up. If they match up, great! You can solve it via the step above. If not, move on to the next step.

Step 2. If you're here, you have a non-exact equation. But all is not lost. First evaluate

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.$$

If this expression is a function of x only, move on to Step 3(a). Remember, divide by what you're subtracting by, and make sure it's a function of only the variable that you're differentiating by.

Otherwise, evaluate

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}.$$

If this expression is a function of y only, go on to step 3(b). You should always divide by what you're substracting by.

If neither one works, you cannot solve the equation using this technique. Get, that, dirt off your shoulder and try again.

Step 3. Find the integrating factor, we have two cases:

Step 3(a). If the expression $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, then an integrating factor is given by

$$\mu(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx\right)$$

Step 3(b). If the expression $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, then an integrating factor is given by

$$\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy\right).$$

Step 4. Multiply the original equation by μ.

Step 4*. Check that your resulting equation is exact, if you have time.

Step 5. Solve the new equation.

Whew. It seems a bit long when written out in full, but it's quite straightforward once you understand what is going on. Let's try it out on a full demonstrative example.

Example 1. Find all the solutions to

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}.$$

We rewrite the equation to get

$$(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0.$$

Hence, $M(x, y) = 3xy + y^2$ and $N(x, y) = x^2 + xy$.

We have

$$\frac{\partial M}{\partial y} = 3x + 2y \text{ and } \frac{\partial B}{\partial x} = 2x + y$$

which shows that the equation is not exact.

Let's find an integrating factor. We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x}.$$

Therefore, an integrating factor $\mu(x)$ exists an is given by

$$\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = e^{\ln(x)} = x.$$

Multiplying through by $\mu(x)$, we get a new equation

$$(3x^2y + xy^2)\,dx + (x^3 + x^2y)\,dy = 0$$

which is exact.

Since we have some time, let's check it.

$$\frac{\partial M'}{\partial y} = (3x^2 + 2xy) \text{ and } \frac{\partial N'}{\partial x} = 3x^2 + 2xy.$$

Exactly! OK, now let's solve as before.

Let's find s solution $F(x, y)$. Consider the system

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2$$
$$\frac{\partial F}{\partial y} = x^3 + x^2y.$$

Let's integrate the first equation. We get

$$\int \frac{\partial F}{\partial x}\,dx = F(x, y) = x^3y + \frac{x^2}{2}y^2 + O(y).$$

where $O(y)$ is some function that only depends on y. Differentiate with respect to y and compare with the second equation of the system of obtain

$$\frac{\partial F}{\partial y} = x^3 + x^2y + O'(y) = x^3 + x^2y.$$

This implies that $O'(y) = 0$, so $O(y) = C$, a constant. Therefore, all the solutions to our differential equation are given by the implicit equation

$$F(x, y) = x^3y + \frac{x^2}{2}y^2 + C.$$

Note that this gives us an *implicit* solution to our differential equation. If your problem asks for a non-implicit solution, make sure to do the extra steps and transform it into the acceptable form. $\diamond$

Remark 2. If you consider the function

$$u(x, y) = \frac{1}{xy(2x + y)},$$

then we get another integrating factor for the same equation. Indeed, you can check that the equation

$$\frac{1}{xy(2x+y)}(3xy + y^2)\, dx + \frac{1}{xy(2x+y)}(x^2 + xy)\, dy = 0$$

is exact.

Hence, the integrating factor may not be unique. Therefore, if you are given an integrating factor (say, on an exam), the only thing you have to do is multiply the equation by the factor and check that the resulting equation is exact.

Remark 3. The above solution was an illustrative example, but also note that the differential equation is a homogeneous equation. Noticing this fact and using the appropriate substitution method tends to yield an quicker solution than the above method. It's because of cases like this that you need to learn which technique is best for your given problem.

4. Difference Equations

These are discrete analogues of autonomous differential equations. In general, we are interested in the long-term behavior of recursions like

$$x_{n+1} = f(x_n).$$

For example, $x_{n+1} = x_n^2 + x_n + 1$. A good way to qualitatively study equations like these is to draw a **staircase diagram** (a.k.a. **cobweb diagram**). Make sure you understand how this works.