MATH 2A RECITATION 10/13/11

1. Introduction

Homeworks are in front, in alphabetical order. Scores are on the back of each sheet, and you can pick them up at the end of class. Graded homeworks from last time are on the second floor next to the math office. You can also pick them up at my office hours. Based on your feedback, my office hours are now **2:00-3:00pm Thursday** after recitation. Also, note that Prof. Ramakrishnan's official office hours are now from Friday 4-5pm. I highly recommend you attend his office hours, because he's awesome.

1.1. Remarks on the last homework.

People seemed to do a good job on exact equations, finding integrating factors, etc.

The only problem was that many people had problems with the first question on the homework. Remember, asymptotic behavior means studying what happens to your solution when your independent variable goes to infinity, e.g. how a solution y behaves as $t \to \infty$.

Make sure you write your analysis in words. We can look at graphs, and you can refer to graphs, and a graph will almost certainly help you find the correct solution (once again, computers make it easier!), but a graph alone is not enough. For instance, many of you had the correct graphs and correctly analyzed dy/dt, but you never said what this behavior meant for y as $t \to \infty$! Remember the same reminders from taking exams: **read the prompt** and **make sure you fully answer the question**. Since we're only grading for the behavior of y, this caught some people off guard.

You guys actually did a pretty good job studying these stability questions, and many probably would have done just as well (or even better) on the homework had we graded for stability. However, we did not grade for that last week, and instead we are focusing on stability on two problems this week. You seem to know how to analyze the asymptotic behavior of differential equations, so I'll focus on the difference equations this week.

2. Difference Equations

Differential equations are great for modeling things with a continually changing population or value. However, if the change happens incrementally rather than continuously, then differential equations are not ideal for modeling. Instead, we will use difference equations, that is, recursively defined sequences, to model such phenomena. Examples of common difference equations include things like monthly compounded interest, seasonal business like hotels in Big Bear, studying populations of things like salmon that spawn once a year, etc.

For the moment, we are restricting ourselves to first-order difference equations, that is, ones that only depend on the last term, like $x_{n+1} = x_n^2$. There are two

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main types of difference equations that come up in the course: linear and nonlinear. We'll cover the most important examples of both. They are also in the book, but in greater abstraction. I find it illustrative to walk through this with a concrete example.

Example 1. Each year, we put 1000 new students into school every year, but due to a zombie infestation/biblical plague/ultra-hard differential equations course only 30% of the students survive and returning the next year. How many students will be in the school in a given year, and what will happen to the population as time goes on?

Solution. This is a linear finite difference equation that we can model with

$$y_{n+1} = .3y_n + 1000.$$

We have

$$\begin{aligned}
y_0 &= 1000 \\
y_1 &= .3y_0 + 1000 \\
y_2 &= .3y_1 + 1000 = .3(.3y_0 + 1000) + 1000 \\
&\cdots \\
y_n &= y_0(1 + .3 + .3^2 + .3^3 + \cdots + .3^{n-1}) + .3^n y_0.
\end{aligned}$$

Now, observe that the first part of y_n is a geometric series, so we have

$$y_n = \frac{y_0(1 - .3^n)}{1 - .3} + .3^n y_0.$$

Therefore, as $n \to \infty$, we see that the limiting population will be $1000/.7 \approx 1429$.

We also see that we can different stable solutions by changing the number of student y_0 that we put in each year. □

More generally, for a linear first-order difference equation

$$y_{n+1} = ry_n + b,$$

the solution is

$$y_n = \frac{b(1 - r^n)}{1 - r} + r^n y_0.$$

Now, let's try an analysis of the most important difference equation.

You might remember that there is a simple differential equation that models exponential growth, namely,

$$\frac{dy}{dt} = ry.$$

As you have probably seen in any math course before this quarter, the solution is

$$y = Pe^{rt}$$

where P is your initial starting population.

However, this model has a major shortcoming: it assumes that populations can increase without bound. Usually there is a limit to the population size based on things like space and food. To handle this, we include a factor that will equal zero when we hit this limit. For example, if the limit is K, then the adjusted growth model is given by

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

Let's study a variation of this with difference equations. This is a very important example, so while it is in the book, I will also go over it in recitation.

Remark 2. The next is a qualitative study of difference equations, like we want you to do for some of the homework problems. However, what I write will probably not suffice for full marks on a solution. On the homework, you need to give more details and analysis.

Example 3. With some work, we can model the above by the finite difference logistics equation

$$u_{n+1} = ru_n(1 - u_n),$$

the one you had on the fourth problem on the homework last week.

We can find the equilibrium by solving

$$u_n = ru_n(1 - u_n).$$

We have

$$\begin{aligned} ru_n - ru_n^2 - u_n &= -ru_n^2 + (r-1)u_n \\ &= u_n(-ru_n + 1 - r). \end{aligned}$$

Therefore, we have equilibrium solutions

$$u_n = 0 \text{ or } u_n = \frac{r-1}{r}.$$

Let's examine the stability of the equilibrium points. Look at values of u_n very close to the equilibrium value (say a tiny fraction away from 0, certainly less than 1). For first point, u_n is much larger than $(u_n)^2$, so our equation can be approximated by

$$u_{n+1} = ru_n(1 - u_n) = ru_n - ru_n^2 \sim ru_n$$

for $|r| < 1$, this converges to zero, and so the first equilibrium point is asymptotically stable in this range.

For the other equilibrium value, write

$$u_n = \frac{r-1}{r} + v_n,$$

so that we get an equilibrium when $v_n = 0$. We can now substitute this into the difference equation to get

$$v_{n+1} = (2 - r)v_n + rv_n^2$$

and chop off the nonlinear term (since v^n is very small) to get

$$v_{n+1} = (2 - r)v_n.$$

This converges to 0 for $|2 - r| < 1$ or $1 < r < 3$. Therefore, the second equilibrium point is asymptotically stable in this range.

At $r = 1$, observe that there is an exchange of stability. What would once converge for one solution will then converge to the other solution.

Now for $r > 3$, the sequence exhibits strange behavior. In particular, for $3 < r < 3.57$ (approximately), the sequence is periodic. However, past this value, we have chaotic behavior, that is, no regularity. Details, pictures, and further analysis are in the book in chapter 2.9.

2.1. **Complex case.** If you are also considering complex solutions like on 2(c) on this week's homework, you need to do analysis for neighborhoods close to 0, not just real intervals. Remember that "closeness" in complex numbers is defined in a different way than the reals, and you might not be able to model this as easily. Namely, if $x + iy \in \mathbf{C}$, the complex numbers, then

$$|x + iy| = \sqrt{x^2 + y^2}.$$

To analyze complex solutions, it is not enough to analyze behavior in a given interval like in the real case. Instead, you must analyze behavior within a small radius of your point.

Remark 4. Problem 2(c) on your homework is actually an easy variation of a very famous work in mathematics. We are not asking you to do the same problem, but maybe reading about this will give you ideas as to how you can analyze your problem. For instance, $z_{n+1} = z_n^2 + c$ where $z_0 = 0$ is the difference equation used to create the **Mandelbrot set**, the first concrete discovery of a fractal! The Mandelbrot set is constructed by studying stability conditions for differing values of c. For instance, $c = 1$ gives the sequence $0, 1, 2, 5, 26, \ldots$ which is not bounded and so 1 is not in the Mandelbrot set. However, if $c = i$, then we get $0, i, (-1 + i), -i, (-1 + i), -i, \ldots$ which is bounded and so i is in the Mandelbrot set.

Now, 2(c) is not the Mandelbrot set, but it hints at that kind of analysis!

Here, you definitely want to use a computer! You can still do all of the analysis by hand, but you'll be missing out on a lot of beautiful mathematics by doing so.

3. Euler's method

People will always call this "Euler's method," but its lesser-used name "tangent line method" is much more descriptive. Here's what's going on.

(1) Start at your initial point, get dx/dt there by evaluating $f(x, t)$.
(2) Go some small distance h along a line with that slope.
(3) Get the actual dx/dt there and go back to step 2.

Repeat this until you reach the t-value t_1 for which you want to know $x(t)$. In fancy math words, you have "approximated the value of $x(t_1)$ with Euler's method with a step size of h." Don't you sound smart saying that?

The recurrence can be stated as follows:

$$t_n = nh$$
$$x_{n+1} = x_n + hf(x_n, t_n)$$

Example 1. Let $dx/dt = f(x_n, t_n) = x - t$, and $x(0) = 0$. Approximate the value of $x(1)$ using Euler's method with a step of size $h = 1/4$.

We set the initial values to $x_0 = 0$, and $t_0 = 0$. Then we have

$$\begin{aligned} t_1 = 1/4 &\quad x_1 = 0 + (1/4)0 = 0 \\ t_2 = 1/2 &\quad x_2 = 0 + (1/4)(-1/4) = -1/16 \\ t_3 = 3/4 &\quad x_3 = -1/16 + (1/4)(-1/16 - 1/2) = -13/64 \\ t_4 = 1 &\quad x_4 = -13/64 + (1/4)(-13/64 - 3/4) = -113/256 \approx 0.44 \end{aligned}$$

A step size of $h = 1/200$ gives us $x(1) = -0.711517$. The actual answer is -0.7182818, that is $(2 - e)$, so the answer isn't too far off.