# MATH 2A RECITATION 10/20/11

## 1. Introduction

Before I begin, I want to make a couple remarks about the midterm. Then I'll do two problems from the book that you will need to do your homework this week. At the end, I'll give some midterm study tips.

1.1. **Midterm Details.** The midterm will be available on **Tuesday, October 25th** and will be due **November 1st at noon**. The midterm review time will take place on Friday at 2-3pm on the first floor of Sloan. I suggest that everybody go, especially if you found the last couple of homeworks difficult. It's always nice to hear material differently, you never know when it will finally click for you.

Please note that **you are not allowed to use my lecture notes on the midterm**. Not at all. You can use your book and notes that you have taken *in class*. That's it. So if you found them helpful, make sure that you glean all the knowledge you can from them while studying.

Today, I'm going to do two problems that are closely related to things that you will have to do on the homework this week. They're actually two parts of one longer problem. They're not hard, but they're intricate and require you to remember a lot of things from previous courses, so pay attention to the details. Review material from last year if any of this is unfamiliar. It's obvious to pay attention to what I am doing, but also pay attention to the things that I am *not* doing. You want to be able to do these problems quickly as well as accurately, and a big part of that is not taking unnecessary steps.

## 2. Problem 7.6.2

We want to find the general solution to the equations given by

$$\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}.$$

We assume a solution of the form $\mathbf{x} = \xi e^{rt}$ thus $r$ and $\xi$ are solutions of

$$\begin{bmatrix} -1-r & -4 \\ 1 & -1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of the coefficients is

$$(-1-r)(-1-r) - 1 \cdot (-4) = r^2 + 2r + 1 + 4 = r^2 + 2r + 5$$

and so by the quadratic formula, the eigenvalues are

$$\frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

The eigenvector corresponding to $-1-2i$ satisfies

$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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*Date*: October 20, 2011.

or $1\xi_1 + 2i\xi_2 = 0$. If $\xi_2 = 1$, then $\xi_1 = -2i$ and so

$$\xi^{(2)} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

and thus the complex-valued solution of the differential equation is

$$x^{(2)}(t) = \begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{(-1-2i)t}.$$

To find real-valued solutions, we take real and imaginary parts, respectively, of $x^{(2)}(t)$. Therefore,

$$\begin{aligned} x^{(2)}(t) &= \begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{-t}(\cos(-2t) + i\sin(-2t)) \\ &= \begin{bmatrix} -2i \\ 1 \end{bmatrix} e^{-t}(\cos(2t) - i\sin(2t)) \\ &= e^{-t} \begin{bmatrix} -2i\cos(2t) - 2\sin(2t) \\ \cos(2t) - i\sin(2t) \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + ie^{-t} \begin{bmatrix} -2\cos(2t) \\ -\sin(2t) \end{bmatrix}. \end{aligned}$$

Hence, the general solution of the differential equation is

$$\mathbf{x} = c_1 e^{-t} \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix}. \diamond$$

*Remark* 1. Stop! Why is this enough? Why do we only need to consider one of the eigenvalues above?

Let's see what happens when we calculate the other eigenvector. The eigenvector corresponding to $-1 + 2i$ satisfies

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

or $1\xi_1 - 2i\xi_2 = 0$. If $\xi_2 = 1$, then $\xi_1 = 2i$ and so

$$\xi^{(1)} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

and thus the complex-valued solution of the differential equation is

$$x^{(1)}(t) = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(-1+2i)t}.$$

To find real-valued solutions, we take real and imaginary parts, respectively, of $x^{(1)}(t)$. Therefore,

$$\begin{aligned} x^{(1)}(t) &= \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{-t}(\cos(2t) + i\sin(2t)) \\ &= e^{-t} \begin{bmatrix} 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + ie^{-t} \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}. \end{aligned}$$

This gives us the same answer as above.

## 3. Problem 7.7.6

This problem is needed to do the last problem on Homework # 4.

We have

$$\mathbf{x'} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$

We want to (a) find a fundamental matrix for the given set of equations, and (b) find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$.

Two linearly independent real-valued solutions of the given differential equation were found in Problem 2 of Section 7.6, which we just did.

Using the result of that problem, we have

$$\Psi(t) = \begin{bmatrix} -2e^{-t}\sin(2t) & 2e^{-t}\cos(2t) \\ e^{-t}\cos(2t) & e^{-t}\sin(2t) \end{bmatrix}.$$

To find $\Phi(t)$, we determine the linear combinations of the columns of $\Psi(t)$ that satisfy the initial conditions $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. So let's plug in $t = 0$!

In the first case, $c_1$ and $c_2$ satisfy

$$\begin{aligned} 0c_1 + 2c_2 &= 1 \\ c_1 + 0c_2 &= 0. \end{aligned}$$

Therefore, $c_1 = 0$ and $c_2 = \frac{1}{2}$

In the second case, $c_1$ and $c_2$ satisfy

$$\begin{aligned} 0c_1 + 2c_2 &= 0 \\ c_1 + 0c_2 &= 1 \end{aligned}$$

so $c_1 = 1$ and $c_2 = 0$. Using these values of $c_1$ and $c_2$ to form the first and second columns of $\Psi(t)$ respectively, we obtain

$$\Phi(t) = \begin{bmatrix} e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \\ \frac{1}{2}e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix}.$$

## 4. Midterm Tips

**Start early.** Learning new math is tough enough as is. Starting early also helps you retain the information better, which means less time that you need to spend studying for your final.

**Make sure you know the essential techniques.** Obvious, but it needs to be said, because without this, you have no hope of doing well on the exam. For example, what's the algorithm for solving a differential equation that can be made exact? You should know this from memory so that you don't waste time on the exam.

**Re-do the homeworks.** Since this is an undergraduate non-major class, the midterm will largely be based on material covered in the homework. Since you theoretically already know how to do all the homework problems, this should go much faster than you think. Focus on the areas in which you are weakest. If you've found the homeworks to be difficult, make sure that clear any conceptual difficulties by asking somebody who knows, like a friend, a TA, or Prof. Ramakrishnan.

**Make up your own midterm.** Try and make up your own midterm, think of what you believe are the most important techniques in the course and think of questions to test your abilities. If you've never done this before, it might seem like a waste of time, but looking back, I've found that writing these practice exams was when I solidified my knowledge about the course.

**Fully answer the question.** I didn't see this so much in this section, but many people missed easy points on the homework because they didn't fully answer the question. Read the question. Then read the question again, slowly. After you write your solution, look at the question again to make sure that you answered every part. Make it easy for your graders, so you don't lose points unnecessarily. Be neat and highlight your answers and key steps.

**Check your answers.** This is incredibly important, especially since the methods in this class require long series of calculations and careful bookkeeping. When you arrive at a solution and have some time left, go back through your steps and see if it makes sense, and do calculations to test your answers as well. You won't get marked off for extra work, but you will lose points if your answer is wrong. For instance, if you've found eigenvalues of a matrix, make sure that your eigenvector is indeed an eigenvector by applying the matrix to it.