MATH 2A RECITATION 11/03/11

1. Announcements

Midterms are available for pickup. Overall, people did pretty well. You can see your score and how you did but we have not yet decided on a grading distribution. Good job! But know that the course, like every other math course, will build upon this knowledge, so make sure that you stay on top of this course. The final will be harder, so don't get complacent.

The methods that we will learn this week aren't particularly tricky, but the calculations do get lengthy, so you want to check your answers and be diligent about your bookkeeping. Since this course is essentially calculation-based, you need to be extra careful.

2. Nonhomogeneous Linear Systems

2.1. **Problem 7.9.4.** We want to find the general solution to

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}.$$

To solve this, we use the method illustrated in Example 1 in Section 7.9.

We first need to find the appropriate transformation matrix corresponding to the given coefficient matrix. We calculate

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} &= (1-\lambda)(-2-\lambda) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2) \end{aligned}$$

so the eigenvalues are $\lambda = -3$ and $\lambda = 2$.

Plugging in $\lambda = -3$, we obtain

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}.$$

Therefore, the -3-eigenspace is spanned by $[1, -4]^T$.

Plugging in $\lambda = 2$, we obtain

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix},$$

so the 2-eigenspace is spanned by $[1, 1]^T$.

Therefore, we have the transformation matrix

$$T = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}.$$

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Inverting T, we find that

$$T^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}.$$

If we let $\mathbf{x} = T\mathbf{y}$ and substitute into the DE, we obtain

$$\begin{aligned} \mathbf{y'} &= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\mathbf{y} + \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{y} + \frac{1}{5}\begin{bmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{bmatrix}. \end{aligned}$$

This corresponds to the two scalar equations

$$\begin{aligned} y_1' + 3y_1 &= (1/5)e^{-2t} + (2/5)e^{t} \\ y_2' - 2y_2 &= (4/5)e^{-2t} - (2/5)e^{t}, \end{aligned}$$

which may be solved by the methods of Section 2.1. For the first equation, the integrating factor is e^{3t} and we obtain

$$(e^{3t}y_1)' = (1/5)e^{t} + (2/5)e^{4t}.$$

Hence,

$$e^{3t}y_1 = (1/5)e^{t} + (1/10)e^{4t} + c_1.$$

For the second equation, the integrating factor is e^{-2t}, so

$$(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}.$$

Hence,

$$e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2.$$

Therefore,

$$\mathbf{y} = \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/10 \\ 2/5 \end{bmatrix} e^{t} + \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{bmatrix}.$$

Finally, multiplying by T, we obtain

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{t} + c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, and the first two terms constitute a particular solution of the nonhomogeneous system.

3. The Matrix Exponential Operator

3.1. **Problem 7.8.19.** We'll do part of this problem, to get an idea of how $\exp(At)$ works and to reminder you of what you need for an inductive proof.

Let

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where λ is an arbitrary real number.

(b) We want to show that $J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$.

Base case: For $n = 1$, we see that

$$J = J^1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

as desired.

Inductive step. Assume that our result holds for n, that is, $J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$. We want to show that it holds for $n+1$, that is, $J^{n+1} = \begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{bmatrix}$. We have

$$\begin{aligned} J^{n+1} = JJ^n &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} \\ &= \begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{bmatrix}. \end{aligned}$$

Hence, by the principle of mathematical induction, every J^k has the desired form.

(c) Recall that the scalar exponential function can be written as a power series:

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

We can define an analogous function by replacing a with an $n \times n$ matrix:

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}.$$

We have

$$\begin{aligned} \exp(Jt) &= I + \sum_{n=1}^{\infty} \frac{J^n t^n}{n!} \\ &= I + \sum_{n=1}^{\infty} \begin{bmatrix} \frac{\lambda^n t^n}{n!} & \frac{n\lambda^{n-1} t^n}{n!} \\ 0 & \frac{\lambda^n t^n}{n!} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} & \sum_{n=1}^{\infty} \frac{\lambda^{n-1} t^n}{(n-1)!} \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \end{aligned}$$

because

$$\sum_{n=1}^{\infty} \frac{\lambda^{n-1} t^n}{(n-1)!} = t\left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}\right) = te^{\lambda t}.$$

(d) Now we will use $\exp(Jt)$ to solve the initial value problem $\mathbf{x}' = J\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

We can write the solution to the initial value problem as $\mathbf{x} = \exp(Jt)\mathbf{x}^0$ by the reasoning in Section 7.7. Therefore, we have

$$\begin{aligned} \mathbf{x} &= \exp(Jt)\mathbf{x}^0 \\ &= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\ &= \begin{bmatrix} x_1^0 e^{\lambda t} + x_2^0 te^{\lambda t} \\ x_2^0 e^{\lambda t} \end{bmatrix} \\ &= \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} e^{\lambda t} + \begin{bmatrix} x_2^0 \\ 0 \end{bmatrix} te^{\lambda t}. \end{aligned}$$