MATH 2A RECITATION 11/10/11

1. Announcements

2. Variation of Parameters

Theorem 1. *Consider the differential equation*

$$y'' + q(t)y' + r(t)y = g(t).$$

Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for the homogeneous equation

$$y'' + q(t)y' + r(t)y = 0.$$

Then a particular solution for the nonhomogeneous differential equation is

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt.$$

Remark 2. Despite what the solution above looks like, it doesn't matter which one is $y_1(t)$ and which one is $y_2(t)$. Like most techniques in this class, it is independent of choice. You'll end up which the same answer either way.

You'll see why there's a $-y_1$ in the formula above for 2nd order linear ODEs once we introduce the formula for general order.

2.1. **How do you use VOP?.**

(a) Check that you have a *linear* ODE.
(b) Check that your highest order term has coefficient 1.
(c) Find a fundamental set of solutions to the associated homogeneous equation. (Denote the complimentary solution by y_c.)
(d) Compute the Wronskian. It should be nonzero.
(e) Plug your set of solutions into the formula to get a particular solution y_p.
(f) If needed, add our result from the formula to the solution from the homogeneous case to get the general solution $y = y_c + y_p$.

Let's see an example.

Example 3. Find a general solution to the differential equation:

$$2y'' + 18y = 6\tan(3t).$$

Solution. We note that we have a linear ODE. Since our formula for variation of parameters requires us to have a coefficient of 1 for the y'' term, we need to divide our equation above by two. Hence, we will be solving the equation

$$y'' + 9y = 3\tan(3t).$$

Suppose we are given the solution for this differential equation:

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t).$$

You can check this this is indeed a solution to our homogeneous equation.

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Therefore, we have

$$y_1(t) = \cos(3t) \text{ and } y_2(t) = \sin(3t).$$

Taking the Wronskian, we see that

$$\begin{aligned} W := W(y_1, y_2) &= \det \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} \\ &= 3\cos^2(3t) + 3\sin^2(3t) = 3. \end{aligned}$$

Therefore, by applying our formula for variation of parameters, we see that the particular solution is

$$\begin{aligned} Y_p(t) &= -\cos(3t) \int \frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t) \int \frac{3\cos(3t)\tan(3t)}{3}\,dt \\ &= -\cos(3t) \int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t) \int \sin(3t)\,dt \\ &= -\cos(3t) \int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t) \int \sin(3t)\,dt \\ &= -\cos(3t) \int [\sec(3t) - \cos(3t)]\,dt + \sin(3t) \int \sin(3t)\,dt \\ &= -\frac{\cos(3t)}{3}(\ln|\sec(3t) + \tan(3t)| - \sin(3t)) + \frac{\sin(3t)}{3}(-\cos(3t)) \\ &= -\frac{\cos(3t)}{3} \ln|\sec(3t) + \tan(3t)|. \end{aligned}$$

Therefore, the general solution is

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln|\sec(3t) + \tan(3t)|.$$

□

Remark 4. Be careful with your bookkeeping for these problems. The formula isn't particularly complicated, but you need to make sure to keep your terms straight. For instance, do not move terms that depend on t from the outside of the integrals to the inside of the integrals.

2.2. **Variation of Parameters for $n = 3$.** We can also use variation of parameters to solve for 3rd order non-homogeneous linear differential equations.

Theorem 5. *Consider a non-homogeneous linear differential equation of order 3:*

$$y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$$

Let $y_1(t), y_2(t), y_3(t)$ be a fundamental set of solutions of the corresponding homogeneous equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0.$$

Then a particular solution to our equation is

$$y_p(t) = \sum_{m=1}^{3} y_m(t) \int \frac{W_m(t)g(t)}{W(t)}$$

where W is the Wronskian and W_m is the determinant of the matrix used to calculate the matrix, but by replacing the mth column by $(0, 0, 1)^t$.

Remark 6. We can similarly define a formula for variation of parameters for non-homogeneous linear ODEs of any order n.

Let's work through an example in full. I will calculate $W_m(t)$ by replacing the mth column with $(0,0,1)^t \cdot g(t)$, because this tends to simplify computations in the example. You can feel free to do this as well, but make sure that in the whole process, you *only multiply by $g(t)$ once*.

Example 7. Solve

$$y''' - 4y' = 12e^{-2t}$$

using variation of parameters.

We first solve the homogeneous equation.

$$r^3 - 4r = r(r^2-4) = r(r+2)(r-2) = 0$$

Therefore, $r = 0, \pm 2$ and so

$$y_c(t) = c_1 + c_2 e^{-2t} + c_3 e^{2t}.$$

Now let's find the particular solution. We have

$$y_1 = 1, y_2 = e^{-2t}, y_3 = e^{2t}.$$

We compute the Wronskian

$$\begin{aligned} W(y_1,y_2,y_3) &= \det \begin{bmatrix} 1 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 0 & 4e^{-2t} & 4e^{2t} \end{bmatrix} \\ &= 1 \det \begin{bmatrix} -2e^{-2t} & 2e^{2t} \\ 4e^{-2t} & 4e^{2t} \end{bmatrix} = -8 - 8 = -16. \end{aligned}$$

Then we compute the associated $W_i(t)$'s:

$$\begin{aligned} W_1 &= \det \begin{bmatrix} 0 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 12e^{-2t} & 4e^{-2t} & 4e^{2t} \end{bmatrix} \\ &= 12e^{-2t} \det \begin{bmatrix} e^{-2t} & e^{2t} \\ -2e^{-2t} & 2e^{2t} \end{bmatrix} \\ &= 12e^{-2t}(2+2) \\ &= 48e^{-2t} \\ W_2 &= \det \begin{bmatrix} 1 & 0 & e^{2t} \\ 0 & 0 & 2e^{2t} \\ 0 & 12e^{-2t} & 4e^{2t} \end{bmatrix} \\ &= 1 \det \begin{bmatrix} 0 & 2e^{2t} \\ 12e^{-2t} & 4e^{2t} \end{bmatrix} \\ &= -24 \\ W_3 &= \det \begin{bmatrix} 1 & e^{-2t} & 0 \\ 0 & -2e^{-2t} & 0 \\ 0 & 4e^{-2t} & 12e^{-2t} \end{bmatrix} \\ &= 1 \det \begin{bmatrix} -2e^{-2t} & 0 \\ 4e^{-2t} & 12e^{-2t} \end{bmatrix} \\ &= 24e^{-4t}. \end{aligned}$$

We then calculate the coefficients of our particular solution:

$$
\begin{aligned}
u_1 &= \int \frac{W_1}{W}\,dt = \int \frac{48e^{-2t}}{-16}\,dt = -3\int e^{-2t}\,dt = -3\cdot\frac{e^{-2t}}{-2} = \frac{3}{2}e^{-2t} \\
u_2 &= \int \frac{W_2}{W}\,dt = \int \frac{-24}{-16}\,dt = \frac{3}{2}t \\
u_3 &= \int \frac{W_3}{W}\,dt = \int \frac{-24e^{-4t}}{-16}\,dt = \frac{3}{2}\int e^{-4t}\,dt = \frac{3}{2}\cdot\frac{e^{-4t}}{-4} = -\frac{3}{8}e^{-4t}.
\end{aligned}
$$

Therefore, our particular solution is

$$
\begin{aligned}
y_p &= u_1y_2 + u_2y_2 + u_3y_3 \\
&= \left(\frac{3e^{-2t}}{2}\right)(1) + \left(\frac{3t}{2}\right)(e^{-2t}) + \left(\frac{-3e^{-4t}}{8}\right)e^{2t} \\
&= \frac{3}{2}e^{-2t} + \frac{3}{2}te^{-2t} - \frac{3}{8}e^{-2t} \\
&= \frac{9}{8}e^{-2t} + \frac{3}{2}te^{-2t}.
\end{aligned}
$$

Now to find our general solution:

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1 + c_2e^{-2t} + c_3e^{-2t} + \frac{9}{8}e^{-2t} + \frac{3}{2}te^{-2t} \\
&= c_1 + c_2e^{-2t} + c_3e^{2t} + \frac{3}{2}te^{-2t}
\end{aligned}
$$

because we absorbed the $\frac{9}{8}$ constant into c_2.