MATH 2A RECITATION 11/17/11

1. Quick Review of Series

The power series of a function $f(x)$ is an expression of the function as an infinite sum:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where a_n and x_0 are numbers, and x is the undetermined variable.

Remark 1. Recall that power series for functions do not always exist. This is an important condition, so it deserves a special name. Indeed, we call a function $f(x)$ *analytic at* x_0 if it has a power series expansion about x_0 that converges near x_0. Translated into math language, we say that "$f(x)$ is analytic at x_0 if its Taylor expansion around x_0 has positive radius of convergence (and converges to $f(x)$)."

Most functions that you will see in this class will be analytic (e.g. trig functions), but you need to be careful.

Now, let's recall one of the most important questions from calculus: When does a power series exist? This is answered by looking at the convergence of the power series.

Definition 2. We say that a power series *converges at* $x = c$ if the series

$$\sum_{n=0}^{\infty} a_n (c - x_0)^n$$

converges. Namely, the series will converge if the limit of partial sums

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (c - x_0)^n$$

exists and is finite.

Note that a power series will always converge if $x = x_0$, in which case the power series is

$$\sum_{n=0}^{\infty} (x_0 - x_0)^n = a_0.$$

Hence, the power series are guaranteed to exist for at least one value of x. This is also why we ask for *positive* radius of convergence for analyticity.

Recall the following fact about power series.

Theorem 3. *Given a power series* $\sum a_n (x - x_0)^n$*, there exists a number* $0 \le \rho \le \infty$*, called the* ***radius of convergence*** *such that the power series converges for* $|x - x_0| < \rho$ *and diverges for* $|x - x_0| > \rho$.

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Remark 4. Note that this doesn't say anything about how the series converges on the boundary. Indeed, almost anything can happen with respect to divergence on the boundary. For instance, there are functions that converge at every point but one on the boundary. However, questions like these will be relegated to more advanced classes.

How do we find the radius of convergence? Again, think back to the tests of convergence from calculus. By slightly modifying those tests, we obtain the following methods for finding ρ.

Theorem 5. *(Ratio test method) Using the ratio test, we can calculate*

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

(Root test method) By the root test, we can calculate

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

Remark 6. It is usually easier to compute the radius of convergence via the ratio test method than the root test, but sometimes you do need to use the root test. If the limit for the ratio test exists, however, the radius of convergence you get is the same as if you used the root test.

Example 7. Calculate the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{n7^{n+1}} (x-5)^n.$$

We use the ratio test method:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-3)^n}{n7^{n+1}} \cdot \frac{(n+1)7^{n+2}}{(-3)^{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{1}{n} \cdot \frac{(n+1)7}{-3} \right| \\ &= \frac{7}{3}. \end{aligned}$$

Why do we care about the radius of convergence? It's because in order for a series solution to a DE to exist at x, it needs to converge at x. Equivalently, if the power series solution does not converge at x, then the series solution will not exist at x. Thus, we see that this information is crucial for solving differential equations.

Now, the key to understanding series solutions is the following almost trivial observation: the only way that

$$a + bx + cx^2 = 0$$

for all x, is for $a = b = c = 0$. Similarly, if we have a power series such that

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$$

for all x, then $a_n = 0$ for all n.

Despite all the fancy tricks that we'll learn with series methods for the rest of the quarter, all of them ultimately trace back to this simple observation.

2. Series Solutions

The philosophy here is as follows: splitting up (expanding) a function into infinitely many simple parts is easier than dealing with the function itself.

2.1. **Ordinary/Singular Points.** Consider the differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

Definition 1. We say that $x = x_0$ is an *ordinary point* if both

$$\frac{q(x)}{p(x)} \text{ and } \frac{r(x)}{p(x)}$$

are *analytic at* $x = x_0$. That is, these two quantities have Taylor series around $x = x_0$.

If a point is not ordinary, it is called a *singular point*.

Remark 2. If $p(x), q(x), r(x)$ are polynomials, then checking that the two quotients above are analytic reduces to saying that $p(x_0) \neq 0$. Since we usually deal with the polynomial case in this class, you can readily use this fact.

2.2. **Series Solution Method.** The basic idea to finding a series solution to a differential equation is to assume that there exists a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

and then to try and determine the coefficients a_n. However, we can only do this if the point $x = x_0$ is ordinary. Thus, we see that the classification above is very important.

Let's see some examples of this in action. I'll do one example slowly, making lots of comments along the way.

Example 3. Determine a series solution for the following differential equation about $x_0 = 0$:

$$y'' + y = 0.$$

Note that $p(x) = 1$ in this case, so every point is ordinary. We're looking for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We need to plug this into our DE, so we calculate the derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Observe that we can renumber these indices to start at $n = 0$, but this is not a good idea here. Generally speaking, if it turns out things become easier by reindexing to start at $n = 0$, we can fix that later.

Now, plug these into our differential equation, so we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

We now want to combine everything into a single series. To do this, we need to get both series starting at the same index, and make sure that the exponents on the x are the same in both series.

A good way to start is to begin by getting matching the exponents of x. It's generally best to get the exponent to be x^n. The second term is already in this form, and we need to shift the first series down by 2 to get the exponent up to this form. By shifting the first power series, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now notice that while shifting, we also got both series starting from the same index. This won't always be the case, but for now, we offer a thanks to the math gods and accept our blessing. We can then add up the two series, to get

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0.$$

Now recall our trivial observation, which implies that

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

for all n. This gives us a **recurrence relation**, like the ones we saw for difference equations. In particular, notice that we always want to include the values for n, since we won't always get a power series whose index starts at $n = 0$.

Now, let's determine the values of the a_n's. Note that the recurrence relation has two different a_n's, so we can't just solve for a_n and get a formula that works for all n. However, we can use the recurrence relation to determine all but two of the a_n's.

First solve the recurrence relation for the a_n with the largest subscript. Doing this, we obtain

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

for $n = 0, 1, 2, \ldots$.

What do we do next? Just start calculating.

$$\begin{aligned}
n = 0, &\quad a_2 = \frac{-a_0}{2 \cdot 1} \\
n = 1, &\quad a_3 = \frac{-a_1}{3 \cdot 2} \\
n = 2, &\quad a_4 = \frac{-a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} \\
n = 3, &\quad a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} \\
n = 4, &\quad a_6 = \frac{-a_4}{6 \cdot 5} = \frac{-a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
n = 5, &\quad a_7 = \frac{-a_5}{7 \cdot 6} = \frac{-a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}.
\end{aligned}$$

Okay, so we're seeing a pattern here: for $k = 1, 2, \ldots$, we have

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \quad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}.$$

Observe that at each step we plugged back in the previous answer, so that when the subscript was even, we could always write a_n in terms of a_0; and when the subscript was odd, we could always write a_n in terms of a_1. This is not always the case.

Furthermore, notice that the formula we developed only works for $k = 1, 2, 3, \ldots$. However, in this particular case, it also works for a_0. Again, this will not always be the case.

You might be worried that we don't know what a_0 and a_1. However, it turns out that we cannot determine them without more information, like an initial value problem.

Now that we have formulas for the a_n's, let's calculate the solution.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} \\
&= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \cdots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^{k+1} a_1}{(2k+1)!} x^{2k+1} + \cdots .
\end{aligned}$$

We then want to collect all the terms with the same coefficient and then factor out the coefficient:

$$\begin{aligned}
y(x) &= a_0 \left[1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \cdots + \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} + \cdots \right] \\
&= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.
\end{aligned}$$

Let's see another example, where I'll skip the waffling.

Example 4. Find a series solution (in terms of powers of x) to the boundary value problem:

$$y'' + 2t^2 y = 0, \quad y'(0) = 0, y(0) = 2.$$

We know the solution is of the form $y = \sum_{i=0}^{\infty} a_i t^i$. Any solution to the problem must have

$$\sum_{i=0}^{\infty} (i+2)(i+1) a_{i+2} t^i + 2 \sum_{i=2}^{\infty} a_{i-2} t^i = 0.$$

Since the coefficients must all cancel out, we must have

$$2a_{i-2} = -(i+2)(i+1) a_{i+2}.$$

Furthermore, we have $a_2 = a_3 = 0$.

From the initial conditions, we know that $a_0 = 2$ and $a_1 = 0$. Therefore, $a_i = 0$ if $a \equiv 0 \pmod 4$, and if $i = 4k$, then we know

$$a_i = \frac{2}{-i(i-1)} a_{i-4} = \frac{2}{-i(i-1)} \cdot \frac{2}{-(i-4)(i-5)} \cdots \frac{2}{-4 \cdot 3} a_0$$

so we have

$$a_{4k} = \frac{(-1)^k 2^{k+1}}{\prod_{j=0}^{k-1}(i-4j)(i-4j-1)}.$$