MA 8, WEEK 10

1. Review Questions

Example 1. Prove that $\sum \frac{1}{p}$ diverges, where the sum extends over all prime numbers.

Proof. Let p_i denote the ith smallest prime number and let $P = \{p_1, p_2, \ldots\}$[1]. Suppose for contradiction that the sum $\sum_{p \in P} \frac{1}{p}$ converges. Then there exists an integer k such that

$$\sum_{i \geq k+1} \frac{1}{p_i} < \frac{1}{2}.$$

Say $p_1, \ldots, p_k$ are small primes, and that $p_{k+1}, p_{k+2}, \ldots$ are large primes. Let N be a positive natural number. Then

$$\sum_{i \geq k+1} \frac{N}{p_i} < \frac{N}{2}.$$

Let N_s be the number of positive integers $n \leq N$ that have only small prime factors and let N_ℓ be the number of positive integers $n \leq N$ that have at least one large prime factor, so

$$N_s + N_\ell = N.$$

The number of positive integers $n \leq N$ divisible by a prime p is $[N/p]$ (also commonly denoted $\lfloor \frac{N}{p} \rfloor$), that is, the largest integer less than N/p. Therefore,

$$N_\ell \leq \sum_{i \geq k+1} \lfloor \frac{N}{p_i} \rfloor \leq \sum_{i \geq k+1} \frac{N}{p_i} < \frac{N}{2}.$$

A number $n \leq N$ with only small prime divisors has the form

$$n = a_n b_n^2$$

where a_n consists of the square-free part of n, so

$$a_n = p_1^{\epsilon_1} \cdots p_k^{\epsilon_k}$$

where ϵ_i is either 0 or 1. Thus, there are precisely 2^k possible square-free parts. Furthermore,

$$b_n \leq \sqrt{n} \leq \sqrt{N}.$$

There are at most $\sqrt{N}$ choices for b_n, so $N_s \leq 2^k \sqrt{N}$.

Choose N so that $2^k \sqrt{N} \leq \frac{N}{2}$ (e.g. $N = 2^{2k+2}$). Then

$$N = N_s + N_\ell < \frac{N}{2} + \frac{N}{2} = N$$

a contradiction. □

[1] However, note that we don't assume a priori that there are infinitely many primes.

Remark 1. You are, of course, familiar with the fact that the harmonic series $\sum_n \frac{1}{n}$ diverges. However, this proof shows that even if you throw out all the composite numbers n from the series, the sum still diverges, in particular the primes forms a still-substantial subset of the positive integers. Contrast with examples like $\sum_n \frac{1}{2^n}$, which we know converges. The distribution of primes is quite mysterious and we still don't understand it completely. For instance, there is an old conjecture called the "twin prime" conjecture, which says that there are infinitely many pair of prime numbers $p, p+2$. Examples of twin primes include (3,5), (11, 13), etc. However, one would expect such phenomena to decrease as the numbers get larger. However, the twin prime conjecture says that despite this, there will always be twin primes as you go further and further out on the number line. There has recently (this year) been a lot of progress on answering this question.

As an aside, note that the above proof also demonstrates (in a cute way) that there are infinitely many prime numbers.

Example 2. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Proof. Since infinitely many coefficients of $\sum a_n z^n$ are nonzero integers, for every N, there is some $n > N$ such that $|a_n| \geq 1$, and so

$$\limsup_{n\to\infty} |a_n| \geq 1.$$

Since $k > 1$ implies that $k^{1/n} > 1$ for any real k, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1.$$

The quantity on the left side is the reciprocal of the radius of convergence R (Hadamard's formula), so

$$\frac{1}{R} \geq 1$$

and hence $1 \geq R$. In other words, the radius of convergence is at most 1. □

Example 3. Suppose that $\{a_n\}$ is a Cauchy sequence in $\mathbf{R}$, and that some subsequence $\{a_{n_\ell}\}$ converges to a point $L \in \mathbf{R}$. Show that the full sequence $\{a_n\}$ also converges to L.

Proof. This is just a slight variant of the fundamental fact that if a sequence converges to a limit, then any subsequence converges to the same limit. However, let's try and walk through a formal proof of this.

Let $\epsilon > 0$. Since $\{a_n\}$ is a Cauchy sequence, there is some N such that for $n, m > M$, we have

$$|a_n - a_m| < \frac{\epsilon}{2}.$$

Since the subsequence $\{a_{n_\ell}\}$ converges to L, there is some M such that $i > M$ implies that

$$|a_{n_i} - L| < \frac{\epsilon}{2}.$$

Therefore, for $i, n > \max(M, N)$, we have

$$|L - a_n| \leq |L - a_{n_i}| + |a_{n_i} - a_n| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $\{a_n\}$ converges to L as $n \to \infty$, as desired. □

Example 4. Suppose that

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants. Show that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the function

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_n x^{n+1}}{n+1}.$$

We have $f(1) = 0$ by our relation between the C_i's. Also, $f(0) = 0$. Since $f(x)$ is a polynomial, $f(x)$ is differentiable with derivative

$$f'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n.$$

By the Mean Value Theorem, since $f(0) = f(1) = 0$, we have $f'(x) = 0$ for some $x \in (0,1)$. Therefore, $\sum C_n x^n$ has a real root in the interval $(0,1)$, as desired. $\square$

Example 5. Suppose that f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Define

$$g(x) = f(x+1) - f(x).$$

Prove that $g(x) = 0$ as $x \to +\infty$.

Proof. Let $\epsilon > 0$. Since $f'(x) \to 0$ as $x \to +\infty$, there exists some N such that if $x > N$, then $|f'(x)| < \epsilon$. We have

$$g(x) = f(x+1) - f(x) = \frac{f(x+1) - f(x)}{(x+1) - x} = f'(t)$$

for some $t \in (x, x+1)$ by the Mean Value Theorem. Since this property holds for all x, choose $x > N$, so that we have $t > x > N$. Now, we have

$$|g(x)| = |f'(t)| < \epsilon,$$

and so $|g(x)| < \epsilon$ for all $x > N$. Therefore, $g(x) \to 0$ as $x \to +\infty$, as desired. $\square$

Example 6. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that f is not integrable on $[a,b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a,b]$. Let $n = |P| - 1$, so n is the number of intervals in the partition P. Every interval will contained at least one rational number and at least one irrational number (e.g. by the pigeonhole principle, since the cardinality of any nonempty open interval is uncountable). Then the $\limsup$ M of f on any interval is 1, while the liminf m of f on any interval is 0, and so

$$U(P,f) = \sum M_i \Delta(x_i) = \sum_1^n 1(x_{i+1} - x_i) = x_{n+1} - x_0 = b - a$$

and

$$L(P,f) = \sum m\Delta(x_i) = \sum_1^n 0(x_{i+1} - x_i) = 0.$$

Since P is an arbitrary partition, these properties holds for all partitions, and therefore,

$$\sup_P L(P,f) = 0, \quad \inf_P U(P,f) = b-a.$$

Hence, f is not integrable. □

Example 7. Let $\{a_n\}$ be the Fibonacci sequence,

$$a_1 = a_2 = 1, \quad a_{n+2} = a_n + a_{n+1}.$$

(1) If $r_n = \frac{a_{n+1}}{a_n}$, then $r_{n+1} = 1 + \frac{1}{r_n}$.

Proof. We have

$$r_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}+a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{r_n}.$$

□

(2) Show that $r = \lim_{n\to\infty} r_n$ exists, and $r = 1 + \frac{1}{r}$. Conclude that $r = (1+\sqrt{5})/2$.

Proof. If $r = \lim_{n\to\infty} r_n$ exists, then

$$r = \lim_{n\to\infty} r_n = \lim_{n\to\infty} 1 + \frac{1}{r_n} = 1 + \frac{1}{\lim_{n\to\infty} r_n} = 1 + \frac{1}{r}$$

and so

$$r = \frac{1+\sqrt{5}}{2}$$

since clearly $r > 0$.

To prove the limit actually exists, note that if $r_n < (1+\sqrt{5})/2$, then $r_n^2 - r_n - 1 < 0$ and so

$$r_n < \frac{2r_n+1}{r_n+1} = r_{n+2}.$$

Thus, $r_1 < r_3 < r_5 < \cdots < 2$ so $\lim_{n\to\infty} r_{2n+1}$ exists. Similarly, $\lim_{n\to\infty} r_{2n}$ exists. Moreover, the equation $r_{n+2} = (2r_n+1)/(r_n+1)$ leads, as before, to the fact that both limits are $(1+\sqrt{5})/2$. □

(3) Show that $\sum_{n=1}^{\infty}$ has radius of convergence $2/(1+\sqrt{5})$.

Proof. The limit

$$\lim_{n\to\infty} \frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}|z| = \frac{1+\sqrt{5}}{2}|z|$$

is < 1 for $|z| < 2/(1+\sqrt{5})$ and > 1 for $|z| > 2/(1+\sqrt{5})$. □

2. More Review Problems

Question 1. *Use Taylor polynomials to approximate* $\sin(0.8)$ *to within* 10^{-4} *of its actual value.*

Solution. Recall that the Taylor series for the function $f(x) = \sin x$ about 0 is

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Let's denote by T_{2n+1} the $(2n+1)$-degree Taylor polynomial of $\sin x$, i.e.

$$T_{2n+1} = \sum_{k=0}^{2n+1}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

As usual, R_{2n+1} will denote the $(2n+1)$-degree remainder term. Hence, $\sin x = T_{2n+1}(x) + R_{2n+1}(x)$. We want to approximate the value of $\sin(0.8)$ to within 10^{-4} of its actual value. Hence, we would like to find n for which $|R_{2n+1}(0.8)| < 10^{-4}$.

By Taylor's theorem we have

$$|R_{2n+1}(x)| = \left|\int_0^x \frac{f^{(2n+2)}(y)}{(2n+1)!}(x-y)^{2n+1}dy\right|$$

Since $f(x) = \sin x$, $|f^{(2n+2)}(y)| \leq 1$ for all values of y. Moreover, for $y \in (0,x)$, we have $(x-y)^{2n+1} \leq x^{2n+1}$. Therefore, we have

$$|R_{2n+1}(x)| \leq \left|\int_0^x \frac{1}{(2n+1)!}x^{2n+1}dy\right| = \frac{x^{2n+2}}{(2n+1)!}$$

Plugging in a few values for n, we find that when we let $n = 3$, $|R_7(0.8)| < 10^{-4}$. This tells us that $T_7(0.8)$ is within 10^{-4} of the actual value of $\sin(0.8)$.

$$T_7(0.8) = 0.8 - \frac{0.8^3}{3!} + \frac{0.8^5}{5!} - \frac{0.8^7}{7!} \approx 0.71736$$

(The actual value of $\sin(0.8)$ is 0.717356) □

Question 2. *Prove Hölder's inequality:*

Theorem 3 (Hölder's inequality)**.** *Let* $p, q > 1$ *be real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. *Then,*

$$\sum_{i=1}^{n}|a_ib_i| \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{q}}$$

Proof. Let $A = \sum_{i=1}^{n}|a_i|^p$ and $B = \sum_{i=1}^{n}|b_i|^q$. If $A = 0$, we must have that $a_1 = \cdots = a_n = 0$. Therefore, both sides of the inequality are 0. Similarly, if $B = 0$, both sides of the inequality are 0 and thus, the inequality is satisfied.

Hence, we can assume that $A \neq 0$ and $B \neq 0$. Let $s = \frac{1}{p}$ and $t = \frac{1}{q}$. Note that $0 < s, t < 1$ and $s + t = 1$. Let's also define

$$x_i = \frac{|a_i|^p}{A} \quad \text{and} \quad y_i = \frac{|b_i|^q}{B}$$

Note that $x_1 + \cdots + x_n = 1$ and that $y_1 + \cdots + y_n = 1$.

We have already shown that $f(x) = e^x$ is a convex function. By the property of convex functions we have

$$f(s\log x_i + t\log y_i) \leq sf(\log x_i) + tf(\log y_i)$$

But the left hand side of the inequality is simply

$$f(s\log x_i + t\log y_i) = e^{s\log x_i + t\log y_i} = e^{s\log x_i}e^{t\log y_i} = x_i^s y_i^t$$

And the right hand side of the inequality is

$$sf(\log x_i) + tf(\log y_i) = se^{\log x_i} + te^{\log y_i} = sx_i + ty_i$$

Therefore, for each i, we have that $x_i^s y_i^t \leq sx_i + ty_i$. Then, summing over all i's we get

$$\sum_{i=1}^n x_i^s y_i^t \leq \sum_{i=1}^n sx_i + ty_i = s\left(\sum x_i\right) + t\left(\sum y_i\right) = s + t = 1$$

But we also have

$$\sum_{i=1}^n x_i^s y_i^t = \sum_{i=1}^n \frac{|a_i|}{A^{1/p}} \frac{|b_i}{B^{1/q}}$$

Therefore, we get

$$\sum_{i=1}^n |a_i b_i| \leq A^{\frac{1}{p}} B^{\frac{1}{q}} = \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{q}}$$

□

Question 4. *Find the Taylor series for* $\log(1+x^6)$ *about 0.*

Solution. Instead of computing derivatives of $\log(1+x^6)$, let's take another (maybe roundabout) approach. In particular, let's first find the Taylor series for $f(x) = \log(1-x)$ about 0. By straightforward computation we get

$$f'(x) = \frac{-1}{1-x} \quad ; \quad f''(x) = \frac{-1}{(1-x)^2} \quad ; \quad f'''(x) = \frac{-2}{(1-x)^3}$$

We can easily see that for $n > 0$, we have

$$f^{(n)}(x) = \frac{-(n-1)!}{(1-x)^n}$$

and thus, $f^{(n)}(0) = -(n-1)!$ for all n. Then, the Taylor series for $\log(1-x)$ about 0 becomes

$$\log(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$$

But noting that $\log(1+x^6)$ is simply $f(-x^6)$, we can use what we just found to compute the Taylor series for $\log(1+x^6)$. More precisely, we have

$$\log(1+x^6) = \sum_{n=1}^{\infty} -\frac{(-x^6)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{6n}}{n}$$

□

3. Analytic functions

Until now, we only studied functions from $\mathbb{R}$ to $\mathbb{R}$. But what happens if we have a function $f : \mathbb{C} \to \mathbb{C}$? One natural question to ask is: what does it mean to differentiate such a function?

Given $f : \mathbb{C} \to \mathbb{C}$, we can write $f(x + iy) = u(x, y) + iv(x, y)$.

Definition 1. We say that f is *analytic* if it satisfies the following Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example 2. $f(z) = e^z$ is an analytic function. Let's first find our $u(x, y)$ and $v(x, y)$.

$$f(x + iy) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Therefore, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Then, the partial derivatives of u and v are:

$$\frac{\partial u}{\partial x} = e^x \cos y \quad ; \quad \frac{\partial u}{\partial y} = -e^x \sin y \quad ; \quad \frac{\partial v}{\partial x} = e^x \sin y \quad ; \quad \frac{\partial v}{\partial y} = e^x \cos y$$

But note that

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

Hence, $f(z)$ satisfies the Cauchy-Riemann equations and thus, $f(z)$ is analytic.

Example 3. Complex conjugation $f(z) = \bar{z}$ is *not* analytic. To see this, we write

$$f(x + iy) = \overline{x + iy} = x - iy$$

Therefore, $u(x, y) = x$ and $v(x, y) = -y$. But then, we have

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

Hence, $f(z) = \bar{z}$ does not satisfy the Cauchy-Riemann equations and thus, $f(z)$ is not analytic.