# MA 8, WEEK 2: SEQUENCES

## 1. Preamble: Why do we bother?

I want to provide some motivation for why we study sequences in the first place. A common belief of many people taking calculus from mathematicians, especially a rigorous proof-based version like Math 1, is that the course mostly consists of difficult proofs of obvious theorems. I mean, we know intuitively what a limit is ("the thing that a point moves towards"), what continuity is ("drawing a graph from left to right without lifting your pencil"), and what a derivative is ("rise over run").

Suppose you believe that these things are obvious over $\mathbf{R}$. Here are a couple of "obvious" theorems you might recall from your high school calculus course.

**Theorem 1.** *(Intermediate value theorem) If $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous, $a < b$ and $f(a) \le 0 \le f(b)$, then there exists a constant $c$ with $a \le c \le b$ such that $f(c) = 0$.*

**Theorem 2.** *(Constant value theorem) If $f : \boldsymbol{R} \to \boldsymbol{R}$ is differentiable and $f'(t) = 0$ for all $t \in \boldsymbol{R}$, then $f$ is constant.*

Now suppose that we replace $\mathbf{R}$ with $\mathbf{Q}$, the rational numbers. The set $\mathbf{Q}$ still satisfies the field axioms and is ordered by $<$, just like $\mathbf{R}$. Suppose that you can give the same "intuitive" definitions of limits, continuity, etc. However, if you continue exploring this strange world, you'll soon realize that things are not as normal as they seem.

**Example 3.** Suppose $f : \mathbf{Q} \to \mathbf{Q}$ is defined by

$$f(x) = \begin{cases} -1, & \text{if } x^2 < 2 \\ 1, & \text{otherwise.} \end{cases}$$

Then (a) $f$ is a continuous function with $f(0) = -1$, $f(2) = 1$, yet there does not exist a $c$ with $f(c) = 0$, and (b) $f$ is a differentiable function with $f'(x) = 0$ for all $x$, but $f$ is not constant.

*Why?* We haven't formally defined what continuity and differentiability mean for $\mathbf{Q}$. However, if you believe that $f$ is discontinuous. Try and find a point at which $f$ is discontinuous.

*Date*: October 8, 2013.

Similarly, if you believe that $f$ is not every differentiable with derivative zero, find a point $x \in \mathbf{Q}$ for which this statement is false. $\square$

You can easily generalize to any number of similar strange functions. A function like the above behaves well and according to our intuition if we consider it as a function from $\mathbf{R}$ to $\mathbf{R}$, but if we pass from $\mathbf{R}$ to $\mathbf{Q}$, which is "almost" $\mathbf{R}$, things don't works like we think they should. Let's see another weird thing that happens, using the same function.

**Example 4.** Consider the function

$$g(t) = t + f(t)$$

where $f(t)$ is defined as above. It turns out that $g'(t) = 1 > 0$ for all $t$, but $g(-8/5) > g(-6/5)$. Therefore, if we work in $\mathbf{Q}$, a function with strictly positive derivative need not be increasing!

The reason for these weird phenomena is that when if replace $\mathbf{R}$ with $\mathbf{Q}$ is that $\mathbf{Q}$ is not *complete*, that is, Cauchy sequences in $\mathbf{Q}$ do not necessary converge in $\mathbf{Q}$ (consider say, a decimal approximation of $\pi$). The completeness of $\mathbf{R}$ is crucial in order for us to do calculus, and to understand this, we need to study sequences.

## 2. Sequences

A **sequence** (of real numbers) is an infinite ordered set of real numbers

$$a_1, a_2, a_3, \ldots$$

indexed by the natural numbers. A convenient way to think of a sequence is as a function (of sets)

$$\begin{aligned} f : \mathbf{N} &\to \mathbf{R} \\ n &\mapsto a_n. \end{aligned}$$

The two definitions are equivalent.

*Remark* 1. If you are still thinking of "functions as formulas," let me disabuse you of that notion right now! A function never *has* to be a formula at all! A general function $f : A \to B$ between sets $A$ and $B$ is *any* rule that assigns to every element $a \in A$ an element $f(b) \in B$.

In Ma 1a, functions are usually assumed to be from $\mathbf{R}$ to $\mathbf{R}$ and *continuous*. Most major theorems about functions will assume this. However, if you expand your notion of "function" to include functions between sets, it will aid your understanding of mathematics. For instance, you can view differentiation (in $x$) as a function

$$\frac{d}{dx} : \{\text{differentiable functions } \mathbf{R} \to \mathbf{R}\} \to \{\text{functions } \mathbf{R} \to \mathbf{R}\}$$
$$f(x) \mapsto f'(x).$$

Studying things in this way allows you to ask interesting questions that you would not have otherwise thought of. For instance, what is the range of the function $\frac{d}{dx}$ above? This links calculus to many other areas in mathematics from linear and abstract algebra to differential equations to logic and computation.

*Remark* 2. Also, answers to the question above are not trivial. For instance, an important function in mathematics and physics is the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

where $z$ is any complex number. This is a sort of "completion" of the factorial function $n!$ in the sense that $\Gamma(n+1) = n!$. A nontrivial result called Hölder's theorem says that $\Gamma(z)$ does not satisfy *any* algebraic differential equation whose coefficients are rational functions. In other words, if you start with rational functions and allow the operations of differentiation, addition/subtraction, and multiplication (by rational functions), as many times as you want, no matter how clever you are, there is no way for you to ever get something that equals $\Gamma(z)$. This is despite the fact that you can define functions like $f(x) = e^x$ via differential equations like

$$f' - f = 0.$$

The main fact that allows for calculus to work is the fact that sequences of real numbers converge if and only if they are Cauchy (intuitively if they "get closer and closer to each other"), as you've seen in lecture.

However, you can state this property of the real numbers is the following even simpler principle, which I learned as the "fundamental axiom of analysis."

**The fundamental axiom of analysis.** "Every bounded increasing sequence has a limit." That is, if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, such that $a_1 \le a_2 \le a_3 \le \cdots$ then there exists an $a \in \mathbf{R}$ such that $a_n \to a$ as $n \to \infty$.

*Remark* 3. This is just a special case of the least upper bound property of the real numbers.

*Remark* 4. Technically, in American English, it would better to call this "nondecreasing" instead of "increasing," but it's slightly less catchy that way. Theorem 10.1 below is the more comprehensive result that you should remember: "bounded monotonic sequences (of reals) converge."

You'll learn a lot of results about sequences, and I'll outline the main ones in the next section. However, they are all different manifestations of the same idea. A unifying principle is that *everything that depends on the fundamental axiom is analysis, everything else is just algebra.* In other words, you can obtain all the theorems of analysis by using the standard "algebraic" properties of the real numbers together with the fundamental axiom.

As an application of this principle, let's get a snappy proof of Apostol Theorem 10.1 via the fundamental axiom.

**Theorem 10.1.** A monotonic sequence converges if and only if it is bounded.

**Lemma 5.** *Every decreasing sequence in* $\boldsymbol{R}$ *bounded below tends to a limit.*

*Proof of Lemma.* If $a_1, a_2, a_3, \ldots$ is a decreasing sequence bounded below, then $-a_1, -a_2, -a_3, \ldots$ is an increasing sequence bounded above. By the fundamental axiom, such a sequence has a limit $L$. By the algebraic properties of sequences (namely, the linearity of convergent series[1]), $-L$ is the limit of $a_1, a_2, \ldots$. □

*Proof of Theorem.* An unbounded sequence cannot converge. Therefore, to prove our result, all we need to show is that a bounded monotonic sequence converges. Monotonic sequences must be nondecreasing or nonincreasing. The fundamental axiom shows that the former case has a limit. The lemma above shows that the latter case has a limit as well. □

## 3. Main Facts about Sequences

(1) **The definition of convergence**: The simplest way to show that a sequence converges is sometimes just to use the definition of convergence, that is, you want to show that for any distance

[1]Apostol I, p.385, equation (10.18).

$\epsilon > 0$, you can force the $a_n$'s to be within $\epsilon$ of our limit, for $n$ sufficiently large.

How should we use the definition? Here's one method.

- Examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on $n$ and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
- Using this simple upper bound, given $\epsilon > 0$, determine a value of $N$ such that whenever $n > N$, our simple bound is less than $\epsilon$. This is usually pretty easy: because these simple bounds go to 0 as n gets large, there's always some value of $N$ such that for any $n > N$, these simple bounds are as small as we want.
- Combine the two above results to show that for any $\epsilon$, you can find a cutoff point $N$ such that for any $n > N$, $|a_n - L| < \epsilon$.

(2) **Arithmetic of sequences**: These tools let you combine previously-studied results to get new ones. Note that only **convergent** sequences behave this well.

- *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
- *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all $n$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

(3) **The Fundamental Axiom**: In $\mathbf{R}$, bounded monotonic sequences have a limit.

(4) **Subsequences and convergence**: if a sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value $L$, all of its subsequences must also converge to $L$.

One particularly useful consequence of this theorem is the following: suppose a sequence $\{a_n\}_{n=1}^{\infty}$ has two distinct subsequences $\{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ that converge to different limits. Then the original sequence cannot converge! This is one of the easiest ways to show that a sequence diverges.

A useful variant that we learned in class is the **Bolzano-Weierstrass theorem**: every bounded sequence has a convergent subsequence.

(5) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value $l$, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to $l$.

(6) **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number $N$ such that for every $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon.$$

You can think of this condition as saying that Cauchy sequences are allowed to vary less and less from the the limit as $n \to \infty$, that is, if you look at points far along enough on a Cauchy sequence, they should be close together.

Why is this useful? Because Cauchy sequences converge, if we just verify the Cauchy condition, we don't need to know the limit explicitly to demonstrate that a sequence converges.

That's a lot to digest at once, but here's two important things to notice. First, almost every result is just some realization or application of the fundamental axiom. Second, due to strong results like (3), (4), and (6), it's easier to show that a sequence converges (without knowing the limit explicitly) rather than finding the limit first and then showing that your sequence converges to it.

## 4. Applications of the Above Facts

**Example 1.** The sequence

$$a_1 = 1,$$
$$a_{n+1} = \sqrt{1 + a_n^2}$$

does not converge.

*Proof.* We proceed by contradiction: in other words, suppose that this sequence does converge to some value $L$, say. Then, consider the limit

$$\lim_{n\to\infty} a_n^2.$$

Since squaring things is a continuous operation,

$$\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n)^2 = L^2.$$

However, we can also use the recursive definition of the $a_n$'s to see that

$$\begin{aligned}\lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \left(\sqrt{1 + a_{n-1}^2}\right)^2 \\ &= \lim_{n\to\infty} (1 + a_{n-1}^2)\end{aligned}$$

However, we know that $\lim_{n\to\infty} a_{n-1}^2 = \lim_{n\to\infty} a_n^2 = L^2$, because the two sequences are the same (just shifted over one place) and thus have the same behavior at infinity. Therefore, we know that both $\lim_{n\to\infty} 1$ and $\lim_{n\to\infty} a_{n-1}^2$ both exist: as a result, we can apply our result on arithmetic and sequences to see that

$$\lim_{n\to\infty}(1 + a_{n-1}^2) = \left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty} a_{n-1}^2\right) = 1 + L^2.$$

So, we've just shown that $L^2 = 1 + L^2$: i.e. $0 = 1$. This is clearly nonsense, so we've arrived at a contradiction. Therefore, our original assumption (that our sequence $\{a_n\}_{n=1}^{\infty}$ converged must be false—in other words, this sequence must diverge. $\square$

**Example 2.** Show that

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

*Solution.* Let's try and prove convergence directly using our method under (1) from above: (a) start with $|a_n - L|$, (b) try to find a simple upper bound on this quantity depending on $n$, and (c) use this simple bound to find for any $\epsilon$ a value of $N$ such that whenever $n > N$, we have

$$|a_n - L| < (\text{simple upper bound}) < \epsilon.$$

Let's look at the quantity $|\sqrt{n+1} - \sqrt{n} - 0|$:

$$\begin{aligned}
|\sqrt{n+1} - \sqrt{n} - 0| &= \sqrt{n+1} - \sqrt{n} \\
&= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{\sqrt{n}}.
\end{aligned}$$

All we did here was hit our $|a_n - L|$ with some algebra, and kept trying things until we got something simple. The specifics aren't as important as the idea here: just start with the $|a_n - L|$ bit, and try everything until it's bounded by something simple and small!

In our specific case, we've acquired the upper bound $\frac{1}{\sqrt{n}}$, which looks rather simple: so let's see if we can use it to find a value of $N$.

Take any $\epsilon > 0$. If we want to make our simple bound $\frac{1}{\sqrt{n}} < \epsilon$, this is equivalent to making $\frac{1}{\epsilon} < \sqrt{n}$, i.e $\frac{1}{\epsilon^2} < n$. So, if we pick $N > \frac{1}{\epsilon^2}$, we know that whenever $n > N$, we have $n > \frac{1}{\epsilon^2}$, and therefore that our simple bound is $< \epsilon$. But this is exactly what we wanted!

In specific, for any $\epsilon > 0$, we've found a $N$ such that for any $n > N$, we have

$$|\sqrt{n+1} - \sqrt{n} - 0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. □

**Example 3.** For any two positive real numbers $x > y > 0$, show that

$$\lim_{n\to\infty} \frac{x^n - y^n}{x^n + y^n} = 1.$$

*Proof.* Using the fact that $0 < y < x$, write $y = cx$, for some positive real number $c < 1$. Then, our limit is just

$$\lim_{n\to\infty} \frac{x^n - (cx)^n}{x^n + (cx)^n} = \lim_{n\to\infty} \frac{x^n - c^n x^n}{x^n + c^n x^n} = \lim_{n\to\infty} \frac{x^n(1 - c^n)}{x^n(1 + c^n)} = \lim_{n\to\infty} \frac{1 - c^n}{1 + c^n}.$$

Now, notice that because $0 < c < 1$, $\lim_{n\to\infty} 1 - c^n = \lim_{n\to\infty} 1 + c^n = 1$. Because of this, we can move our limit above into the fraction (because both the top and bottom limits exist,) and get

$$\lim_{n\to\infty} \frac{1 - c^n}{1 + c^n} = \frac{\lim_{n\to\infty} 1 - c^n}{\lim_{n\to\infty} 1 + c^n} = \frac{1}{1} = 1.$$

So our original limit is 1. □

**Example 4.** Let

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Then the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

*Solution.* Let's try showing that this sequence is monotonically increasing and bounded to prove it converges.

*Monotonically increasing*: this is not hard to show: because the difference between $a_{n+1}$ and $a_n$ is

$$a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!},$$

which is positive, we know that $a_{n+1} > a_n$ for every $n$.

*Bounded*: This is not much harder. Take any term $a_n$, and expand it as the sum

$$a_n = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n}.$$

How can we make this simpler, into something we can easily study and show is finite? One way is to simply take the denominators of all of these fractions and replace all of the numbers greater than 2 with 2's. In other words, notice that

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} + \ldots + \frac{1}{2 \cdot 2 \cdot \ldots \cdot 2} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}. \end{aligned}$$

But we know the sum on the right! (Or at least recognize the pattern.) In particular, by remembering our geometric sum identities from whenever they came up in high school (or proving them via induction, if you've forgotten them), we have

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}} = \frac{2^{n-1} - 1}{2^{n-1}} < 1.$$

So the entire sum is bounded above by $1 + 1 + 1 = 3$, for any $n$! So it's bounded above and monotonically increasing, and therefore convergent, via our theorem. □

**Example 5.** The sequence

$$\{a_n\}_{n=1}^{\infty} = 0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

diverges.

*Proof.* Both

$$0, 0, 0, 0, 0, 0, \ldots$$

and

$$1, 1, 1, 1, 1, 1, \ldots$$

are subsequences of $\{a_n\}_{n=1}^{\infty}$. Therefore, because the first subsequence converges to 0 and the second subsequences converges to 1, which are distinct values, our tool tells us that the original sequence $\{a_n\}_{n=1}^{\infty}$ cannot converge, and thus must diverge. □

*Remark* 6. Note that this illustrates what the Bolzano-Weierstrass theorem doesn't tell us. Since $a_n$ is bounded, we know that there is at least one convergent subsequence. However, it doesn't tell us what the subsequence converges to, or whether different subsequences have different limits!

**Example 7.** Show that the sequence

$$a_n = \frac{1}{n}$$

converges.

*Solution.* There are a number of ways to show that this sequence converges. Let's try and show that $a_n$ satisfies the Cauchy condition. Let $\epsilon > 0$. We want to find an $N$ such that if $n, m > N$, then $|a_n - a_m| < \epsilon$.

*Remark* 8. Note that $N$ must depend on $\epsilon$ except in the most trivial examples, like constant sequences.

Note that if $n, m > N$, then $a_n = \frac{1}{n}, a_m = \frac{1}{m} < \frac{1}{N}$. By the triangle inequality,

$$|a_n - a_m| \leq |a_n| + |a_m| < \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

Since we want this last quantity to be $\leq \epsilon$, we see that any number $N > \frac{2}{\epsilon}$ should work.

Let's write this up as a clean proof.

*Proof.* Let $\epsilon > 0$. Pick a natural number $N > \frac{2}{\epsilon}$. If $n, m > N$, then $a_n = \frac{1}{n}, a_m = \frac{1}{m} < \frac{1}{N}$. By the triangle inequality,

$$|a_n - a_m| \leq |a_n| + |a_m| < \frac{1}{N} + \frac{1}{N} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $\{a_n\}$ is a Cauchy sequence, and so converges. $\square$