MA 8, WEEK 3: SERIES, POWER SERIES, LIMITS, CONTINUITY

1. Series

To every sequence $\{a_n\}_{n=1}^{\infty}$, we can associate a **series**

$$(*) \qquad \sum_{k=1}^{\infty} a_k.$$

In other words, a series is just a sum of infinitely many terms.

Remark 1. A series usually consist of infinitely many nonzero terms, but it can consist of many terms that are zero. For instance, suppose that $a_1 = 1$ and $a_2 = 2$ and $a_n = 0$ for $n \geq 3$. Then

$$\sum_{n=1}^{\infty} a_n = 1 + 2 = 3.$$

Don't forget about this possibility!

We say that a series $\sum_{n=1}^{\infty} a_n$ **converges** or **diverges** if the sequence of *partial sums* $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ converges or diverges, respectively.

Remark 2. Note this difference! The "obvious" way to turn a sequence into a series (as in $(*)$) is *not* the same as the sequence of partial sums that we check to see that a series converges or diverges.

We've seen these results and examples in lecture, so let's just briefly recall the tools we have to prove such things.

If our sequences consist only of *positive* terms:

- **Comparison test:** If $\{a_n\}$ and $\{b_n\}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then
$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$
This is best used if you can find a simple upper bound for the sequence that converges (like $\sum 1/n^2$) or a simple lower bound that diverges (like $\sum 1/n$).
- **Ratio test:** If $\{a_n\}$ is a sequence of positive numbers such that $a_n \neq 0$ for all sufficiently large n and
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$
then we have the following three possibilities:
  - If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Date: October 15, 2013.

– If $r = 1$, then the test doesn't say anything it could either converge or diverge.

This is best used when you have something that grows like a geometric series or better, like 2^n or $n!$.

If our sequences can also include *negative* terms:

- **Alternating series test:** If $\{a_n\}$ is a sequence such that
  (1) the a_n's alternate in sign and
  (2) $\lim_{n\to\infty} |a_n| = 0$ monotonically
  then the series $\sum_{n=1}^{\infty} a_n$ converges.
  Try this first whenever you see an alternating series.
- **Absolute convergence ⇒ convergence:** If $\{a_n\}$ is a sequence such that
$$\sum_{n=1}^{\infty} |a_n|$$
  converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.
  Whenever you have a sequence with both negative and positive terms that is not alternating, you'll want to use this. Since most tests require the fact that your sequence is positive, you'll commonly apply this test to a sequence and then use some other test on the sequence $\{|a_n|\}$.
- **Root test:** Suppose that
$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$
  From here, the results are like that of the ratio test. If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $L > 1$, then the series diverges. If $L = 1$, we again can't say anything.
- **nth Term test:** We know that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by the Cauchy condition. We can often use the contrapositive of this to show divergence, namely, if $\lim_{n\to\infty} A_n \neq 0$ or if the limit does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

What happens if you rearrange terms in a series? Before we move on from generalities about series, I have to say something about the following phenomenon, which displays some of the subtleties between sequences and series.

Think, for a moment about what series **are**: just infinite sums of things! Recall that addition is **commutative**, i.e. the order in which we add things up doesn't matter! In other words if we have a finite sum, we know that

$$1 + 2 + 3 = 3 + 2 + 1 = 6.$$

A natural question we could ask, then, is the following: does this also work if we're summing infinitely many terms, i.e. for a series? In other words, if we rearrange the terms in the series (say)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

will it still sum up to the same thing?

Let's see what happens. Specifically, consider the following way to rearrange our series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \\ &=^{?} \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots\end{aligned}$$

In the second rearrangement, we've ordered terms in the following groups:

$$\dots + \left(\frac{1}{\text{odd number}} - \frac{1}{2 \cdot \text{that odd number}} - \frac{1}{2 \cdot \text{that odd number} + 2}\right) + \dots$$

Notice that every term from our original series shows up in exactly one of these groups. Specifically, each odd number clearly shows up once: as well, for any even number, there are two cases: either it has exactly one factor of 2, in which case it's of the form (2·an odd number) and shows up exactly once, or it's a multiple of 4, in which case it shows up as a (2·an odd number+2), and also shows up once. So this is in fact a proper rearrangement! We haven't forgotten any terms, nor have we repeated any terms.

But, if we group terms as indicated below in our rearrangement, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &=^{?} \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \dots \\ &= \frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots\right) \\ &= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.\end{aligned}$$

So: if rearranging terms doesn't change the sum of an infinite series, we've just shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

The only number that is equal to half of itself is 0: therefore, this series must sum to 0!

However: look at our series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ again. In specific, if we just expand this sum without rearranging anything, we can see that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \dots \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \dots \\ &= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \frac{1}{9 \cdot 10} + \dots \\ &\geq \frac{1}{2}.\end{aligned}$$

So this sum definitely **cannot** converge to 0, as all of its partial sums are $\geq \frac{1}{2}$!

What's going on here? The reason this happens is that the series above is at sort of the "boundary" between absolute convergence and divergence. It's not absolutely convergent, but yet it's not divergent either!

Well, we have an answer to our question about rearranging series fairly definitively: we've just shown that rearranging terms in a series can be very unpredictable, and do very strange things.

In fact, we have the following two theorems about what happens when series are rearranged:

Theorem 3. *Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Then rearranging the a_n's does not change what our series converges to.*

Theorem 4. *Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series that is not absolutely convergent. Then for any $r \in \mathbb{R} \cup \{\pm\infty\}$, we can rearrange the a_n's so that the resulting series converges to r.*

2. Power Series

A kind of series that we are particularly interested in are what are called **power series** (in $(z - a)$) or **power series expanded about** a, which are series of the form

$$p(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + \dots + a_n(z-a)^n + \cdots$$

where a is some number (usually real, but can be complex). Often a is taken to be 0 (as in lecture).

Why are power series important? There are many reasons, but one reason is that power series expansions at given points are related the notion of a function being **analytic**, which is a very desirable property of functions, especially once we get to differentiation.

Here's the main result we have about power series.

Theorem 1. *(Apostol 11.7) Suppose that the power series $\sum a_n z^n$ converges for at least one $z \neq 0$ (a complex variable), say, $z = z_1$ and that it diverges for at least one z, say $z = z_2$. Then there exists a real number $r > 0$ such that the series converges absolutely if $|z| < r$ and diverges if $|z| > r$.*

This $r > 0$ is called the **radius of convergence**. For the name radius of convergence to make sense, you need to think of z as a complex number.

Power series will converge inside the radius of convergence, and diverge out of it, but the behavior on the boundary (i.e. on precisely the radius of convergence) can be almost anything. For instance, consider the following examples (in each, z is a complex number).

- The series can always converge on its boundary $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.
- The series can never converge on its boundary: $\sum_{n=1}^{\infty} z^n$.
- The series diverges at precisely one points along its boundary $\sum_{n=1}^{\infty} \frac{z^n}{n}$.
- The series diverges at a finite number of points along its boundary (add some multiples of the last example together).

Example 2. Find the radius of convergence and the interval of convergence (real numbers at which the series converges) of the series

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n-1}}.$$

Proof. We begin with the ratio test for the absolute convergence. Let $a_n = \frac{n(x+2)^n}{5^{n-1}}$. Then

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(x+2)^{n+1}}{5^n} \cdot \frac{5^{n-1}}{n(x+2)^n}\right| \\ &= \frac{n+1}{5n}|x+2| \to \frac{|x+2|}{5} \end{aligned}$$

as $n \to \infty$. Hence, the power series converges absolutely if $\frac{|x+2|}{5} < 1$, so the radius of convergence is $r = 5$. Hence, the potential interval of convergence is $-5 \leq x + 2 \leq 5$, that is, $-7 \leq x \leq 3$.

Let's check the convergence at the boundary points. Since we don't expect you to be fully comfortable with all properties of complex numbers, let's stick to checking it at the real boundary points.

At $x = -7$, we have

$$\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5n(-1)^n.$$

Since $\lim_{n\to\infty} 5n(-1)^n \neq 0$, this series does not converge by the nth term test.

At $x = 3$, we have

$$\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^{n-1}} = \sum_{n=1}^{\infty}.$$

Clearly this series diverges, e.g. by the nth term test again. Hence, the series converges on the interval $(-7, 3) \subset \mathbf{R}$. □

3. Limits and Continuity

Definition 1. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x\to a} f(x) = L$$

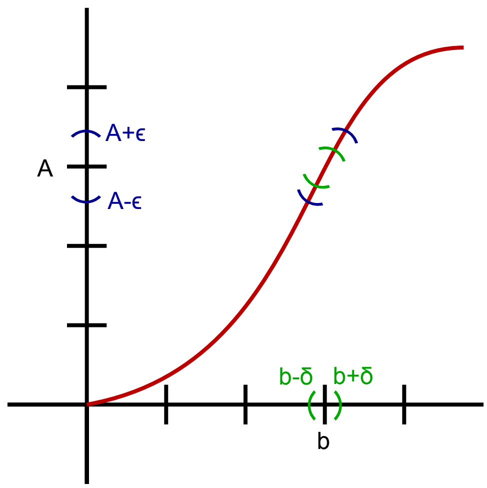
if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition 2. A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

This definition is a little strange, isn't it? At least, the $\epsilon - \delta$ "rigorous" definition is somewhat strange: how do these weird symbols connect with the rather simple concept of "as x approaches a, $f(x)$ approaches $f(a)$"? To see this a bit better, consider the following image:



This graph shows pictorially what's going on in our "rigorous" definition of limits and continuity: essentially, to rigorously say that "as x approaches a, $f(x)$ approaches $f(a)$", we are saying that

- for any distance ϵ around $f(a)$ that we'd like to keep our function,
- there is a neighborhood $(a - \delta, a + \delta)$ around a such that
- if f takes only values within this neighborhood $(a-\delta, a+\delta)$, it stays within ϵ of $f(a)$.

Basically, what this definition says is that if you pick values of x sufficiently close to a, the resulting $f(x)$'s will be as close as you want to be to $f(a)$ – i.e. that "as x approaches a, $f(x)$ approaches $f(a)$."

This, hopefully, illustrates what our definition is trying to capture – a concrete notion of something like convergence for *functions*, instead of sequences of real numbers. So: how can we prove that a function f has some given limit L? Motivated by this analogy to sequences, we have the following method to prove that $\lim_{x \to a} f(x) = L$ straight from the definition:

(1) First, examine the quantity

$$|f(x) - L|.$$

Using algebra/cleverness, try to find a simple upper bound for this quantity of the form

$$(\text{things bounded when } x \text{ is near } a) \cdot (\text{function based on} |x - a|).$$

Some sample candidates: things like $|x - a| \cdot (\text{constants})$, or $|x - a|^3 \cdot (\text{bounded functions like } \sin(x))$.

(2) Take your bounded part, and **bound** it! In other words, find a constant bound $C > 0$ and a value $\delta_1 > 0$ such that whenever x is within δ_1 of a, we have

$$\text{(bounded things)} < C.$$

(3) Take your function based on $|x-a|$ and your constant C from the above step, and starting from the equation

$$\text{(function based on } |x-a|) < \frac{\epsilon}{C},$$

solve for $|x-a|$ in terms of ϵ and C, by performing only reversible steps. This then gives you some equation of the form

$$|x-a| < (\text{thing in terms of } C, \epsilon\text{'s}).$$

Define δ_2 to be this "thing in terms of C, ϵ's."

(4) Let $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have just proven that we satisfy both the equations

$$\text{(bounded things)} < C, \qquad \text{and}$$
$$\text{(function in } |x-a|) < \frac{\epsilon}{C}.$$

If we combine these observations with the simple bound we derived in our first step, we've proven that whenever $|x-a| < \delta$, we have

$$|f(x) - L| < \text{(bounded things)}(|x-a| \text{ things}) < C \cdot \frac{\epsilon}{C} = \epsilon.$$

But this is exactly what we wanted to prove – this is the $\epsilon - \delta$ definiton of a limit! So we are done.

The following example ought to illustrate what we're talking about here:

Example 3. The function $\frac{1}{x^2}$ is continuous at every point $a \neq 0$.

Proof. We want to prove that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$, for any $a \neq 0$.

We proceed according to our blueprint:

(1) First, we examine the quantity $\left|\frac{1}{x^2} - \frac{1}{a^2}\right|$:

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \left|\frac{a^2}{a^2x^2} - \frac{x^2}{a^2x^2}\right| \\ &= \left|\frac{a^2 - x^2}{a^2x^2}\right| \\ &= \left|\frac{(a-x)(a+x)}{a^2x^2}\right| \\ &= |a-x| \cdot \left|\frac{(a+x)}{a^2x^2}\right| \\ &= |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right|.\end{aligned}$$

By algebraic simplification, we've broken our expression into two parts: one of which is $|x-a|$, and the other of which is bounded near $x = a$.

For values of x rather close to a, because $a \neq 0$, we can bound this as follows: pick x such that x is within $a/2$ of a. Then we have

$$\begin{aligned}\left|\frac{(a+x)}{a^2x^2}\right| &\leq \left|\frac{(a+(3a/2))}{a^2x^2}\right| \\ &\leq \left|\frac{(a+(3a/2))}{a^2(a/2)^2}\right| \\ &= \left|\frac{10}{a^3}\right|\end{aligned}$$

which is some nicely bounded constant. So, when we pick our δ, if we just make sure that $\delta < a/2$, we know that we have this quite simple and excellent upper bound

$$\left|\frac{(a+x)}{a^2x^2}\right| < \left|\frac{10}{a^3}\right|.$$

(2) So: we have bounded the bounded part by $\left|\frac{10}{a^3}\right|$. Now, we want to take the remaining $|x-a|$ part, which is exactly $|x-a|$, and solve the equation

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}$$

for $|x-a|$, given any arbitrary $\epsilon > 0$. Conveniently, this is already done! In fact, if we're using our blueprint and we can make our "function in terms of $|x-a|$" precisely $|x-a|$, this is always this easy. Therefore, if we set $\delta_2 = \frac{a^3\epsilon}{10}$, then whenever $|x-a| < \delta_2$, we have

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}.$$

(3) Now, set $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have

$$|f(x) - L| < |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right| < \frac{10}{a^3} \cdot \frac{a^3\epsilon}{10} = \epsilon,$$

which is precisely what we needed to show to satisfy the $\epsilon - \delta$ definition of a limit. Therefore, we have proven that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$ for any $a \neq 0$, as desired.

$\square$

4. Three Results on Continuity

Limits and continuity are wonderfully useful concepts, but working with them straight from the definitions can be a little cumbersome, as we saw above. As a result, we often apply the following theorems to allow us to prove that certain limits exist without going through the definition every time.

Theorem 1. *(Squeeze theorem:) If f, g, h are functions defined on some interval $I \setminus \{a\}$[1] such that*

$$\begin{aligned} f(x) \le g(x) \le h(x), \forall x \in I \setminus \{a\}, \\ \lim_{x\to a} f(x) = \lim_{x\to a} h(x), \end{aligned}$$

then $\lim_{x\to a} g(x)$ exists, and is equal to the other two limits $\lim_{x\to a} f(x), \lim_{x\to a} h(x)$.

Example 2.

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Proof. So: for all $x \in \mathbb{R}, x \neq 0$, we have that

$$\begin{aligned} -1 &\le \sin(1/x) \le 1 \\ \Rightarrow -x^2 &\le x^2 \sin(1/x) \le x^2; \end{aligned}$$

thus, by the squeeze theorem, as the limit as $x \to 0$ of both $-x^2$ and x^2 is 0,

$$\lim_{x\to 0} x^2 \sin(1/x) = 0$$

as well. □

Theorem 3. *(Limits and arithmetic): if f and g are functions such that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist, then we have the following equalities:*

$$\begin{aligned} \lim_{x\to a} (\alpha f(x) + \beta g(x)) &= \alpha \left(\lim_{x\to a} f(x)\right) + \beta \left(\lim_{x\to a} g(x)\right) \\ \lim_{x\to a} (f(x) \cdot g(x)) &= \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right) \\ \lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) &= \left(\lim_{x\to a} f(x)\right) / \left(\lim_{x\to a} g(x)\right), \text{ if } \lim_{x\to a} g(x) \neq 0. \end{aligned}$$

Corollary 4. *Every polynomial is continuous everywhere.*

Proof. To start, we know that the functions $f(x) = x$ and $f(x) = 1$ are trivially continuous. By multiplying these functions together and scaling by constant factors, we can create any polynomial; thus, by the above theorem, we know that any polynomial must be continuous, as we can create it from continuous things through arithmetical operations. □

Theorem 5. *(Limits and composition): if $f : Y \to Z$ is a function such that $\lim_{y\to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x\to b} g(x) = a$, then*

$$\lim_{x\to b} f(g(x)) = L.$$

Specifically, if both functions are continuous, their composition is continuous.

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[1] The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol \, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

Example 6.

$$\lim_{x \to a} \sin(1/x^2) = \sin(1/a^2),$$

if $a \neq 0$.

Proof. By our work earlier in this lecture, $1/x^2$ is continuous at any value of $a \neq 0$, and from class $\sin(x)$ is continuous everywhere: thus, we have that their composition, $\sin(1/a^2)$, is continuous wherever $x \neq 0$. Thus,

$$\lim_{x \to a} \sin(1/x^2) = \sin(1/a^2),$$

as desired. □

5. Discontinuity

How do we show a function is discontinuous?

Just like the easy way to show that a sequence diverges, we want to show that there are two sequences that converge to the same number in the domain, but do not converge to the same limit in the range after applying the function to the sequences.

Lemma 1. *For any function $f : X \to Y$, we know that $\lim_{x \to a} f(x) \neq L$ iff there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:*

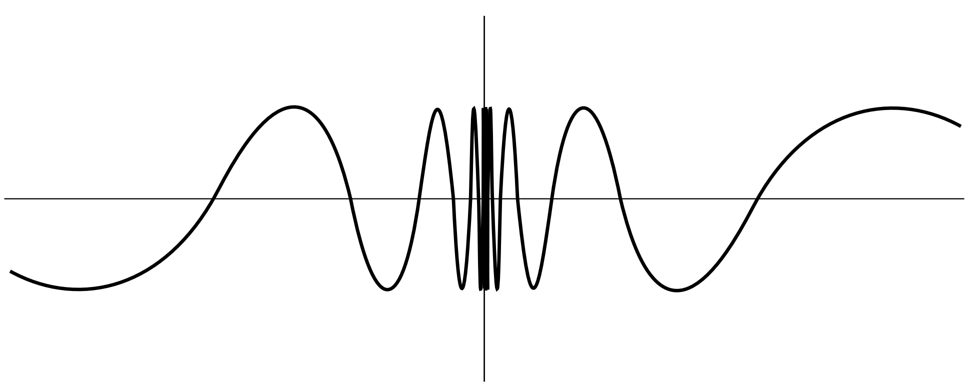
- $\lim_{n \to \infty} a_n = x$, *and*
- $\lim_{n \to \infty} f(a_n) \neq L$, *and*

This lemma makes proving that a function f is discontinuous at some point a remarkably easy: all we have to do is find a sequence $\{a_n\}_{n=1}^{\infty}$ that converges to a on which the values $f(a_n)$ fail to converge to $f(a)$. Basically, it allows us to work in the world of sequences instead of that of continuity. This makes a lot of our calculations easier to make.

We can see this in the following example.

Proposition 2. *The function* $\sin(1/x)$ *has no defined limit at 0.*

Proof. So: before we start, consider the graph of $\sin(1/x)$:



Visual inspection of this graph makes it clear that $\sin(1/x)$ cannot have a limit as x approaches 0; but let's rigorously prove this using our lemma, so we have an idea of how to do this in general.

So: we know that $\sin\left(\frac{4k+1}{2}\pi\right) = 1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+1)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+1)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+1)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+1}{2}\pi\right) = \lim_{k\to\infty} 1 = 1$,

our lemma says that if $\sin(1/x)$ has a limit at 0, it must be 1.

However: we also know that $\sin\left(\frac{4k+3}{2}\pi\right) = -1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+3)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+3)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+3)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+3}{2}\pi\right) = \lim_{k\to\infty} -1 = -1$,

our lemma **also** says that if $\sin(1/x)$ has a limit at 0, it must be -1. Thus, because $-1 \neq 1$, we have that the limit $\lim_{x\to 0} \sin(1/x)$ cannot exist, as desired. □

6. One-Sided Limits

Let's conclude with something fairly elementary: the concept of a **one-sided limit**.

Definition 1. For a function $f : X \to Y$, we say that

$$\lim_{x\to a^+} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x\to a^-} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x < a) \Rightarrow (|f(x) - L| < \epsilon).$$

Basically, this is just our original definition of a limit except we're only looking at x-values on one side of the limit point a: hence the name "one-sided limit." Thus, our methods for calculating these limits are pretty much identical to the methods we introduced before: we work one example below, just to reinforce what we're doing here.

Proposition 2.

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1.$$

Proof. First, examine the quantity

$$\frac{|x|}{x}.$$

For $x > 0$, we have that

$$\frac{|x|}{x} = 1;$$

therefore, for any $\epsilon > 0$, it doesn't even matter what δ we pick! – because for any x with $0 < x$, we have that

$$\left| \frac{|x|}{x} - 1 \right| = 0 < \epsilon.$$

Thus, the limit as $\frac{|x|}{x}$ approaches 0 from the right hand side is 1, as desired. $\square$

One-sided limits are particularly useful when we're discussing limits at infinity, as we describe in the next section.

7. Limits at Infinity

Definition 1. For a function $f : X \to Y$, we say that

$$\lim_{x \to +\infty} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to -\infty} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x < N) \Rightarrow (|f(x) - L| < \epsilon).$$

In class, we described a rather useful trick for calculating limits at infinity:

Proposition 2. *For any function $f : X \to Y$,*

$$\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right).$$

Similarly,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right).$$

The use of this theorem is that it translates limits at infinity (which can be somewhat complex to examine) into limits at 0, which can be in some sense a lot easier to deal with: as opposed to worrying about what a function does at extremely large values, we can just consider what a different function does at rather small values (which can make our lives often a lot easier.)