MA 8, WEEK 5: MIDTERM PRACTICE PROBLEMS

1. Proof Techniques

Basically, you are all—as a class—quite capable with proof methods; so there's not a lot to say here. As you already know it often amounts to checking the definitions, conditions of the theorems or a combination of these.

Two proof techniques are, however, worth mentioning: proof by contradiction and induction.

1.1. **Proof by Contradiction.**

We want to prove something like "statement P implies statement Q". To prove this by contradiction, we follow the following steps:

- Step 1: Assume P is true.
- Step 2: Assume "not Q" is true.
- Step 3: Reach a contradiction.

Example 1. Show that the difference of any rational number and any irrational number is irrational.

Proof. What are the statements "P" and "Q" in this case? We can rewrite the question as follows: "If x is a rational number and y is an irrational number, then $x - y$ is an irrational number". Now, it's easier to see what the statements P and Q are. Namely, P is the statement "x is rational and y is irrational"; and Q is the statement $x - y$ is irrational.

Let's write up a formal proof using contradiction now!

(Step 1:) Suppose x is rational and y is irrational.

(Step 2:) Suppose $x - y$ is rational.

(Step 3:) Since x and $x - y$ are rational, we can find integers a, b, c, d such that $x = \frac{a}{b}$ and $x - y = \frac{c}{d}$. Note that we now have:

$$\begin{array}{rrcl} & \frac{a}{b} - y & = & x - y = \frac{c}{d} \\ \Leftrightarrow & y & = & \frac{a}{b} - \frac{c}{d} \\ \Leftrightarrow & y & = & \frac{ad-bc}{bd} \end{array}$$

However, $ad - bc$ and bd are both integers. Hence, y is also rational, which contradicts our hypothesis (in Step 1) that y is irrational.

Therefore, the difference of any rational number and any irrational number is irrational. □

1.2. **Proofs by Induction.**

Suppose that you have a claim $P(n)$ – a sentence like "$2^n \geq n$", for example. How do we prove that this kind of thing holds by induction? Well: we generally follow the outline below: $P(n)$ holds, for all $n \geq k$.

- Step 1: we prove (by hand) that $P(k)$ holds, for a few base cases.
- Step 2: Assuming that $P(m)$ holds for all $k \leq m < n$, prove that $P(n)$ holds. (This step is called the induction hypothesis).

Date: October 29, 2013.

- Step 3: $P(n)$ holds for all $n \geq k$.

Example 2. Suppose that $a_1, \dots a_n$ are all real numbers from the interval $[0, 1]$. Show that

$$\prod_{i=1}^{n}(1-a_i) \geq 1 - \sum_{i=1}^{n} a_i.$$

Proof. We proceed by induction. Our base case is trivial: for $n = 1$, our claim is

$$\prod_{i=1}^{1}(1-a_i) = 1 - a_1 \geq 1 - a_1 = 1 - \sum_{i=1}^{1} a_i,$$

which is true.

Inductive step: assume that our claim holds for n. We seek to prove our claim for $n+1$: i.e. take any collection of $n+1$ values $a_1, \dots a_{n+1}$. Examine the product

$$\prod_{i=1}^{n+1}(1-a_i) = (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right).$$

If we apply our inductive hypothesis to $(\prod_{i=1}^{n}(1-a_i))$, we can see that

$$\begin{aligned}
\prod_{i=1}^{n+1}(1-a_i) &= (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right) \\
&\geq (1-a_{n+1}) \cdot \left(1 - \sum_{i=1}^{n} a_i\right) \\
&= 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) + a_{n+1} \cdot \left(\sum_{i=1}^{n} a_i\right) \\
&\geq 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) \\
&= 1 - \sum_{i=1}^{n+1} a_i.
\end{aligned}$$

Notice that we used that all of the a_i's were ≤ 1 when we plugged in our inductive hypothesis (as otherwise $(1 - a_{n+1})$ would have been negative, and our inequality would have gotten reversed.) Also notice that we used that all of the $a_i's$ were positive when we said that $a_{n+1} \cdot (\sum_{i=1}^{n} a_i) \geq 0$ and therefore getting rid of it only makes the right-hand-side smaller. □

Warning 3. Induction is used to prove things about *finite* values of n. In other words, when proving statements about the limit of a sequence, you need to be careful with induction. Even if a statement is true for all a_n, it might not be true for the limit $\lim a_n$.

Let's see how it can go wrong.

Consider this sequence:

$$a_1 = 1 \quad , \quad a_{n+1} = \frac{a_n}{n+1}$$

We can show that $a_n > 0$ for all n:

The base case is simple: $a_1 = 1 > 0$. Now assume $a_n > 0$ for all $n \leq N$. Then,

$$a_{N+1} = \frac{a_N}{N+1} > 0$$

since a_N and $N+1$ are both positive numbers. So by induction, we have shown that $a_n > 0$ for all n.

However, this does NOT imply that $\lim_{n\to\infty} a_n > 0$. Looking back at our sequence, we can rewrite a_n as

$$a_n = \frac{1}{n!}$$

where $n!$, called n factorial, is the product of all the integers from 1 to n.

It's easy to see that a_n actually converges to 0 (for example squeeze between the constant sequence $b_n = 0$ and the sequence $c_n = \frac{1}{n}$). Hence, $\lim_{n\to\infty} a_n = 0 \not> 0$.

2. Sequences

What tools do we have for showing that a sequence converges?

(1) **Definition of convergence**: We can simply go ahead and check using the definition of convergence. In case you have forgotten:

$\lim_{n\to\infty} a_n = \lambda$ if for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - \lambda| < \varepsilon$.

(2) **Monotone and Bounded Sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges. This is a useful trick if you've ran into a sequence that you have no idea where it converges to, but just need to show that it goes *somewhere*.

(3) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l. Basically, whenever you're studying a sequence whose terms are really complex and messed-up, use the squeeze theorem to bound it above and below by something simple you understand – i.e. if you were looking at $a_n = n^{-2}\sin(n)$, you could bound this above and below by $\pm n^{-2}$.

(4) **Cauchy sequences** A sequence is Cauchy if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m, n \geq N$

$$|a_m - a_n| < \epsilon.$$

We also saw that a sequence converges if and only if it is Cauchy!

Example 1. Show that the sequence $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges.

Proof. We will show that this sequence is monotone and bounded.

Claim: The sequence is monotone increasing.

First, let's rewrite the sequence as $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n}$. We will use induction to prove the claim. The base case is obvious since $a_1 = 1 < a_2 = \sqrt{2}$.

Now, suppose $a_n < a_{n+1}$ for all $n \leq N$. Then, we have:

$$a_N^2 = (\sqrt{2a_{N-1}})^2 = 2a_{N-1} < 2a_N = (\sqrt{2a_N})^2 = a_{N+1}^2$$

Note that our induction hypothesis implies in particular that $a_n > a_1 > 0$ for all $n \leq N$. Hence,

$$a_N^2 < a_{N+1}^2 \quad \Leftrightarrow a_N < a_{N+2}$$

Therefore, the sequence is monotone increasing.

Claim: The sequence is bounded above by 2.

We again use induction (see how useful induction is?!!) to prove the claim. Base case, as always (or often), is obvious: $a_1 = 1 < 2$.

Now, assume $a_n < 2$ for all $n \leq N$. Then, we have

$$a_{N+1} = \sqrt{2a_N} < \sqrt{2 \cdot 2} = \sqrt{4} = 2$$

Therefore, the sequence is bounded above by 2.

So far, we have shown that the sequence is monotone increasing, and that it is bounded above by 2. Therefore, the sequence converges! □

Example 2. Let x, y be a pair of positive real numbers such that $x < y$. Show that

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = y.$$

Proof. This time, let's apply the squeeze theorem. Specifically, notice that

$$y = (y^n)^{1/n} < (x^n + y^n)^{1/n} < (y^n + y^n)^{1/n} = 2^{1/n}y.$$

Therefore, because

$$\lim_{n\to\infty} y = y$$

and

$$\lim_{n\to\infty} 2^{1/n}y = y \cdot \lim_{n\to\infty} 2^{1/n} = y \cdot 1 = y,$$

the squeeze theorem tells us that

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = y$$

as well. □

3. Limits and Continuity

Recall that a function f is continuous at x if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

In other words, a function f is continuous at x if $\lim_{y\to x} f(y)$ exists and is equal to $f(x)$.

Example 1 (Thomae's Function a.k.a the popcorn function)**.** Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are relatively prime integers with } q > 0 \\ 0 & x \text{ is irrational} \end{cases}$$

Show that $f(x)$ is continuous at every irrational number and discontinuous at every rational number.

Proof. The question is asking us to prove two things: 1. when x is irrational, f is continuous at x; and 2. when x is rational, f is *not* continuous at x. Let's first prove that f is continuous at irrational numbers.

Let x be irrational. Then, we first note that $f(x) = 0$.

Now, let $\varepsilon > 0$. Choose a natural number m large enough so that $\frac{1}{m} < \varepsilon$. Then, x lies in a unique interval of the form $\left(\frac{k}{m}, \frac{k+1}{m}\right)$, say $x \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$.

Note that for all $n \leq m$, there can be at most 1 number of the form $\frac{r}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$. Why? Suppose $\frac{r}{n}, \frac{s}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$. Then, we have

$$\begin{array}{rrcl} & \left|\frac{r}{n} - \frac{s}{n}\right| & < & \left|\frac{K+1}{m} - \frac{K}{m}\right| \\ \Rightarrow & \frac{|r-s|}{n} & < & \frac{1}{m} \\ \Rightarrow & \frac{1}{n} & < & \frac{1}{m} \\ \Rightarrow & n & > & m \end{array}$$

But this contradicts our assumption that $n \leq m$. Hence, there is at most one number of the form $\frac{r}{n} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$.

In particular, this implies that there are finitely many rational numbers $\frac{p}{q} \in \left(\frac{K}{m}, \frac{K+1}{m}\right)$ in the reduced form where $q \leq m$ (We showed above that, in fact, there can be at most m such numbers). Denote these numbers $p_1, \ldots, p_s$ and let

$$\delta = \min(|x - p_1|, \ldots, |x - p_s|)$$

We claim that for all y with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. We can divide this in two cases: Y can be irrational or rational.

Suppose y is an irrational number with $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |0 - 0| = 0 < \varepsilon$$

Now suppose y is a rational number with $|x - y| < \delta$. Since y is rational, we can write $y = \frac{p}{q}$. But since y is closer to x than any of $p_1, \ldots, p_s$, we conclude $q > m$ (we chose $\delta = \min(|x-p_1|, \ldots, |x-p_s|)$ precisely to make this happen!). Therefore, we have the following

$$|f(x) - f(y)| = \left|0 - \frac{1}{q}\right| = \frac{1}{q} < \frac{1}{m} < \varepsilon$$

Combining these two cases, we see that $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$. Hence, f is continuous at x.

Now, let's prove that f is not continuous at every rational numbers.

Let $x = \frac{p}{q}$ be a rational number. Then, $f(x) = \frac{1}{q}$. To show that f is *not* continuous at x, we have to find *some* value of $\varepsilon > 0$ for which we cannot find a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Consider $\varepsilon = \frac{1}{q}$. I claim that for this value of ε, we can't find a value of δ satisfying the conditions of the definition.

Let's prove this by contradiction. Suppose there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. However, note that since $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains an irrational number, say $y \in (x - \delta, x + \delta)$. Now, for this y we have $|x - y| < \delta$. However, since y is irrational,

$$|f(x) - f(y)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q} \not< \varepsilon = \frac{1}{q}$$

This is a contradiction! Hence, we conclude that for $\varepsilon = \frac{1}{q}$, there does not exist $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Therefore, f is not continuous at every rational number. $\square$

4. More Practice: Warmups

Example 1. Suppose that $x, y \in \mathbf{R}$ have the property that $|x - y| < \epsilon$ for every $\epsilon > 0$. Show that $x = y$.

Proof. Suppose $x \neq y$. Then $|x - y| = a > 0$. We have a positive real number b such that $a > b > 0$. Since $b > 0$, by assumption $|x - y| < b$. But then we have a contradiction since this implies that

$$a = |x - y| < b.$$

Hence, we must have $x = y$. $\square$

Example 2. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $x_n \leq y_n$ for all n. Prove, using the definition of convergence, that if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $x \leq y$.

Proof. This is essentially a "finite" version of the infinite squeeze theorem proved in class. Recall the proof there. Is it possible to translate the proof there to here?

One way to think about this is to use the arithmetic of sequences, whence we see that the sequence $\{y_n - x_n\}$ is convergent with limit $y - x$. However, how can you show that $y - x$ is nonnegative?

Here's one way to do this. It's not the quickest or slickest way, but it works with minimal cleverness. Suppose for contradiction that $x > y$. Consider $\epsilon = \frac{y-x}{2} > 0$. Since $x_n \to x$ and $y_n \to y$, there exists an N such that for all $n > N$, we have

$$|x_n - x| + |y_n - y| < \epsilon + \epsilon = y - x.$$

By using the triangle inequality, we have

$$|x_n - x| + |y_n - y| \leq |x_n - y_n + y - x|.$$

We know that $y - x$ is positive, and by our deduction above, we must have $|x_n - y_n + y - x| < y - x$. Hence, $x_n - y_n < 0$, that is, $y_n > x_n$, but this is impossible, since it contradicts our assumption on x_n and y_n. $\square$

5. Theory Practice Questions

Example 1. Let $\alpha(n)$ denote the function that takes in any natural number n and returns the number of prime factors it has, counting repeated primes. For example, $\alpha(6) = 2, \alpha(8) = 3, \alpha(120) = 5, \alpha(17) = 1$. Let $\alpha(0) = 0$, to avoid any possibly ambiguous cases. Show that the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^3}$$

converges.

Proof. To do this, simply notice that $\alpha(n) < n$ for any n, as the number of prime factors in a number can never exceed that number. (Why? Well, suppose that you have a number with k prime factors. Then this number is at least 2^k, because each prime factor contributes a multiplier of at least 2, the smallest prime. We know that $k < 2^k$, for all k, so this proves our claim.)

Using this, notice that we have

$$\frac{\alpha(n)}{n^3} < \frac{n}{n^3} = \frac{1}{n^2},$$

for all $n \in \mathbb{N}$: applying the comparison theorem then tells us that because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so must our series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^3}.$$

□

Example 2. Let $\sum a_n$ be a convergent series of non-negative numbers. If $p > 1$, show that $\sum a_n^p$ converges. (Note that we can't use the arithmetic of convergent sequences here (Why?).)

Proof. Since the series $\sum a_n$ of non-negative terms converges, then $\lim a_n = 0$. Therefore, there exists an N such that for all $n \geq N$, we have $0 \leq a_n < 1$. Recall that for any number a such that $0 \leq a < 1$ is such that $0 \leq a^p < a$ for all $p > 1$. Therefore, for all $n \geq N$, we have $0 \leq a_n^p < a_n$. Since the convergence of $\sum a_n$ implies convergence of $\sum a_{N+n}$ because $\sum_{n=0}^{N} a_n$ is a finite sum. We have

$$\sum a_{N+n}^p < \sum a_{N+n}$$

so by the comparison test, $\sum a_{N+n}^p$ converges. Therefore, $\sum a_n^p$ converges as well. □

Example 3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with the property that $f(x) \in \mathbf{R}\backslash\mathbf{Q}$ for all $x \in \mathbf{R}$. Show that f must be constant.

Proof. Suppose for contradiction that f is not constant, so there exists an $a, b \in \mathbf{R}$ such that $f(a) < f(b)$. Between any two irrational numbers, there is a rational number c such that $f(a) < c < f(b)$. By the Intermediate Value Theorem, there exists a number r such that $f(r) = c$. However, this is a contradiction since $f(r)$ must be irrational. Hence, f must be constant. □

Remark 4. Note that a similar proof works if you assumed that $f(x)$ was always rational, instead of irrational.

Example 5. Let $I = [0, 1]$ be the closed unit interval in $\mathbf{R}$. Suppose that f is a continuous function $f : I \to I$. Show that $f(x) = x$ for at least one $x \in I$.

Proof. Consider the function

$$g(x) = x - f(x)$$

which is also continuous. If $g(x) = 0$ for any $x \in I$, we have proven our result. Suppose for contradiction that $g(x) \neq 0$ for all $x \in I$.

Since $f(0) \in [0, 1]$, we have

$$g(0) = 0 - f(0) \leq 0,$$

and since we are assuming $g(0) \neq 0$, then $g(0) < 0$. Also, since $f(1) \in [0, 1]$, we have

$$g(1) = 1 - f(1) \geq 0$$

and since $g(1) \neq 0$, we have $g(1) > 0$. By the Intermediate Value Theorem, there is some $x \in (0, 1)$ such that $g(x) = 0$, a contradiction. □

Example 6. Let

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

Is $f(x)$ differentiable at 0?

Proof. We simply use the definition of the derivative: i.e. $f(x)$ is differentiable at 0 if and only if the limit

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

exists. Because $f(0) = 0$, this is just the limit

$$\lim_{h\to 0} \frac{f(h)}{h}.$$

Dealing with this directly is hard. However, for any h, because $|f(h)| \le h^2$, we can bound this quantity above by $\frac{h^2}{|h|}$ and below by $-\frac{h^2}{|h|}$. We can see that both of these quantities (after simplifying them to $\pm|x|$) go to 0 as x goes to 0; therefore, using the squeeze theorem, we can finally note that our limit

$$\lim_{h\to 0} \frac{f(h)}{h} = 0$$

as well. Therefore the derivative exists (because this limit exists!), and is specifically 0. □

Example 7. Let $f : \mathbf{R} \to \mathbf{R}$ and suppose that

$$|f(x) - f(y)| \le (x - y)^2$$

for all $x, y \in \mathbf{R}$. Show that f is constant.

Proof. Suppose that $f(x) \ne f(y)$ for some x and y, and say $x < y$. Take a positive integer n such that

$$|f(x) - f(y)| > \frac{(x - y)^2}{n}.$$

(Such an n exists by the Archimedean property of the real numbers.) Now, partition the closed interval $[x, y]$ into n intervals of equal length with endpoints

$$x_0 = x, x_1 = x + \frac{y - x}{n}, \ldots, x_n = y.$$

Then by the assumption on our function

$$|f(x_i) - f(x_{i-1})| \le (x_i - x_{i-1})^2 = \left(\frac{x - y}{n}\right)^2 = \frac{(x - y)^2}{n^2}.$$

Hence,

$$|f(y) - f(x)| \le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le n \cdot \frac{(x - y)^2}{n^2} = \frac{(x - y)^2}{n} < |f(y) - f(x)|,$$

which is a contradiction. □

Proof. Here's an alternative proof. For any $x \in \mathbf{R}$, we have

$$|f'(x)| = \lim_{h\to 0} \frac{|f(x + h) - f(x)|}{|h|} \le \lim_{h\to 0} \frac{h^2}{|h|} = 0.$$

Therefore, f is constant. □

Example 8. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous. Suppose that f has the property that if U is an open interval of $\mathbf{R}$ (possibly of infinite length, and $\mathbf{R}$ itself is one), then the image $f(U)$ is also an open interval. Show that f is a strictly monotone function, that is, either $f(x) < f(y)$ for all $x < y$ or $f(x) > f(y)$ for all $x < y$.

Proof. Suppose that f were not monotone. Then there exist real numbers $x < y < z$ such that either (1) $f(x) < f(y)$ and $f(z) < f(y)$ or (2) $f(x) > f(y)$ and $f(z) > f(y)$. We can assume without loss of generality that (1) holds (otherwise, we could replace f by $-f$).

Since f is continuous, by the extreme value theorem, f attains its maximal value on the closed interval $[x, y]$, that is, there exists an $m \in [x, y]$ such that $f(m) \geq f(t)$ for all $t \in [x, y]$. By the inequalities on (1), we must have $m \neq x$ and $m \neq y$. Therefore, $m \in (x, y)$ and $f(m)$ is the maximal value of f on (x, y). However, this implies that the image of (x, y) under f is not an open interval, since there is no neighborhood of $f(m)$ that lies in that image, which contradicts our assumption on f. Hence, f is strictly monotone. □

Example 9. If two continuous functions are equal over the rationals, then they are always equal. In other words, if $f(r) = g(r)$ for all $r \in \mathbf{Q}$, then $f(x) = g(x)$ for all $x \in \mathbf{R}$.

Proof. It is first useful to prove the following preliminary lemma.

Lemma 10. *If $f(a) \neq g(a)$ for some $a \in \mathbf{R}$, then there is a number $\delta > 0$ such that $f(x) \neq g(x)$ whenever $|x - a| < \delta$.*

Proof of Lemma. Consider the function

$$h(x) := f(x) - g(x),$$

noting that h is continuous and $h(a) \neq 0$. Set $\epsilon = \frac{|h(a)|}{2}$. Since $\epsilon > 0$, by using the continuity of h, we can find a $\delta > 0$ such that $|h(x) - h(a)| < \epsilon$ whenever $|x - a| < \delta$. Now, if $|x - a| < \delta$, then

$$|h(x) - h(a)| < \epsilon = \frac{|h(a)|}{2}$$

so $|h(x)| > |h(a)|/2$ and in particular, $h(x) > 0$. However, this implies that when $|x - a| < \delta$, then $h(x) \neq 0$ and so $f(x) \neq g(x)$. □

We proceed by contradiction. Suppose that $f(r) = g(r)$ for every $r \in \mathbf{Q}$ and let a be a real number such that $f(a) \neq g(a)$. Using our lemma above, we obtain a $\delta > 0$ such that $g(x) \neq g(x)$ whenever $|x - a| < \delta$. Between any two real numbers, there is a rational number, so find an $r \in \mathbf{Q}$ such that $a - \delta < r < a + \delta$. However, then $|r - a| < \delta$ and so $f(r) \neq g(r)$, but $f(r) = g(r)$ as $r \in \mathbf{Q}$. This is a contradiction, hence, $f(a) \neq g(a)$ cannot be true. Hence, for all $a \in \mathbf{R}$, we have $f(a) = g(a)$. □

Example 11. If f is differentiable at a, then it is continuous at a.

Proof. Recall that f is differentiable at a if f is defined in a neighborhood of a and

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. If this limit exists, we call it $f'(a)$.

If f is differentiable at a, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + (x-a)\frac{f(x)-f(a)}{x-a} \\ &= f(a) + \left(\lim_{x\to a} x - a\right)\left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right) \\ &= f(a) + 0 \cdot f'(a) \\ &= f(a).\end{aligned}$$

Hence, by the definition of continuity, f is continuous at a. $\square$