MA 8, WEEK 7: FUNDAMENTAL THEOREM OF CALCULUS, TAYLOR SERIES AND APPROXIMATION

1. More Mysteries of R: Transcendental Numbers

We saw last week that there are countably infinitely many rational numbers and uncountably many irrational numbers. However, it turns out that irrational numbers are more intractable than they might seem, which tells us that we (as mathematicians, as the scientific community) are not looking at the irrational numbers in the right way. But let's see one more reason why irrationals are hard to understand.

Let's try and look at the real numbers in another way. Say a number in $\mathbf{R}$ is **algebraic** if it is the root of a polynomial equation with coefficients in the rational numbers. Obviously, all rational numbers are algebraic, since any such number $a \in \mathbf{Q}$ is the root of the polynomial

$$x - a,$$

which has rational coefficients. However, there are some irrational numbers that are also algebraic. For instance, $\sqrt{2}$ is algebraic since it is the root of

$$x^2 - 2.$$

Roughly speaking, the algebraic numbers represent the real numbers you can get by starting with the integers and adding, subtracting, multiplying, dividing, and taking roots. The algebraic numbers also have nice properties, for instance, the sum of two algebraic numbers is algebraic, and the product of two algebraic numbers is also algebraic (Try and work these out). This leads one to ask, are there any real numbers that are not algebraic, for which there is absolutely no polynomial equation for which it is a root?

Surprisingly, the answer turns out to be yes. Such non-algebraic numbers are called **transcendental**. The first transcendental number to be found was

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.1100010000000000\cdots$$

by Liouville. It's fun to try and understand why the number above cannot satisfy any possible polynomial equation (Hint: one way is to think about the growth of the sequence and how to account for it as a polynomial). It turns out that π and e are also transcendental (as you might have expected), but the proofs of these are much more involved (look them up!). In general, proving that numbers are transcendental is really hard. For instance, it is a long-standing problem to determine whether sums like $e + \pi$ are transcendental. They are expected to be, but there has not yet been a valid proof of such a result. Such lack of progress indicates that we do not really understand transcendental numbers very well.

However, even without such explicit examples, using what we proved before, we can show that despite the fact we have few known examples of transcendental numbers, "most" real numbers are transcendental. The reason is that the rationals

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are countably infinite, so the polynomials you can form with rational coefficients are also countably infinite (e.g. we could count them by increasing degree). Since algebraic numbers must be roots of polynomials, and each polynomial only has finite many roots, and there are only countably infinitely many polynomials, we conclude that there are only countably infinitely many algebraic numbers. Since $\mathbf{R}$ is uncountably infinite, this implies that except for a countably infinite set, $\mathbf{R}$ consists of transcendental numbers; that is, there are uncountably many transcendental numbers. Indeed, one can easily show that there are just as many transcendental numbers as real numbers!

Even though there are so many transcendental numbers, we still don't know good ways to construct them. For instance, a famous question regarding these numbers from the beginning of the 20th century asks:

> Q: If a is an algebraic number that is not 0 or 1, and b is an irrational algebraic numbers, is a^b necessarily transcendental?

The answer turns out to be be yes (a theorem of Gelfond-Schneider), but this took a considerable amount of thinking to work out. Let's try and briefly think like a mathematician and try to remove hypothesis, find the converse, strengthen conclusions, etc.

What happens if you remove the adjective "algebraic" on a and b? Well, we can then get things like

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2.$$

(By the Gelfond-Schneider Theorem, $a = 2^{\sqrt{2}}$ is transcendental, not algebraic.) Similarly, if $a = 3$ and $b = (\log 2)/(\log 3)$, which is transcendental, then $a^b = 2$ is algebraic.

Note that the result only works in one direction. It's an open question to find necessary *and* sufficient conditions on a and b such that a^b is algebraic.

2. Fundamental Theorem of Calculus

Here's the main result of the course. Essentially everything else in single-variable calculus is just a corollary of this result. This is the precise sense in which "integration and differentiation are inverse operations."

Theorem 1. *(The Fundamental Theorem of Calculus, Version 1) Let F be a continuous function on the interval $I = [a, b]$. Suppose F is differentiable everywhere in the interior of the interval I (that is, on (a, b)) with derivative f, which is (Riemann) integrable. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

As you may remember from your previous calculus course, we usually say that F is the **antiderivative** or the **primitive** of f.

Let's take a moment and just marvel at the miracle of this.

Why are all of these conditions needed?

Remark 2. Note that we can only talk about differentiability on the *interior* of I, because in $\mathbf{R}$, the definition of the derivative at a point depends on limits coming from both the positive and negative sides.

We also commonly apply the following version of the Fundamental Theorem of Calculus. The following is a slightly souped-up version of the FTC presented in class, and comes from your textbook.

Theorem 3. *(The Fundamental Theorem of Calculus, Version 2) Let f be a function that is integrable on $[a, x]$ for each x in $[a, b]$. Let c be such that $a \le c \le b$ and define a new function*

$$F(x) = \int_c^x f(t)\,dt, \quad \text{if } a \le x \le b.$$

Then the derivative $F'(x)$ exists at each point x in the open interval (a, b) where f is continuous, and for such x, we have

$$F'(x) = f(x).$$

As a corollary of this second theorem, we obtain the following useful result that is also sometimes referred to as a "Fundamental Theorem of Calculus."

Corollary 4. *Suppose that f is continuous on an open interval I, and let F be any antiderivative of f on I. Then, for each c and each x in I, we have*

$$F(x) = F(c) + \int_c^x f(t)\,dt.$$

This might seem stupid at first, since it's just saying the same thing as the theorems above. However, we want to interpret this in the following way. When we apply this, we are really interested in the value of the antiderivative $F(x)$. This says that you can determine the value of $F(x)$ at a random point precisely by evaluating it at some point $F(c)$ and calculating the integral from x to c. This is useful, for instance, if, say $F(c)$ is easy to calculate for a specific c, but $F(x)$ is generally hard to compute.

2.1. **The interaction between differentiability and integration.** So we have sort of a dictionary between the two worlds for nice functions: the "light world" of differentiability, and the "dark world" of integration, with the bridge between them given by the Fundamental Theorem of Calculus.

It's important to understand what actions in one world look like when transported to the other world via these bridges. This should be something you should cultivate as you continue to learn calculus.

Consider two of the main tools we have for differentiation:

- The chain rule, which says that for differentiable f, g, we have $(f(g(x))' = f'(g(x)) \cdot g'(x)$.
- The product rule, which says that for differentiable f, g, we have $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$.

What do these look like in the "dark world" of integration?

Taking the product rule over to the dark side, we obtain the following technique.

Theorem 5. *(Integration by Parts) If f, g are a pair of C^1-functions on $[a, b]$, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Of course, you probably learned this as a technique in calculus symbolically as

$$\int u\,dv = uv - \int v\,du.$$

I like to this of this result "physically" as saying that you can switch the function to which you are applying $\frac{d}{dx}$ inside an integral, as long as you switch the sign and add an error term. It's cool to try and do this for some basic mechanics questions, for instance.

Taking the chain rule over to the dark side, we obtain another useful technique.

Theorem 6. *(Change of Variables/Substitution) If f is a continuous function on $g([a,b])$ and g is a C^1-function on $[a,b]$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Remark 7. Don't forget to change the endpoints.

Again, you can interpret this as follows. Your $g(x)$ is a sort of "distortion" of the interval, so to account for this, you need to integrate by $g'(x)\,dx$ instead of just dx itself.

This kind of geometric reasoning may not be so useful in a one-dimensional setting, but you will see that it will be crucial to understand in multivariable calculus, when identifying symmetries and decomposing a function into a composite of simpler functions greatly simplifies calculations.

3. Applications of the Fundamental Theorem of Calculus

Let's start with some familiar examples.

Example 1. We have

$$\int_0^b x^p\,dx = \frac{b^{p+1}}{p+1}$$

Proof. Note that $f(x) = x^p$ is continuous and bounded on $[0b]$ for any b. Furthermore, we know that

$$\left(\frac{x^{p+1}}{p+1}\right)' = \frac{p+1}{p+1}x^p = x^p$$

for all x, so $\frac{x^{p+1}}{p+1}$ is a primitive of x^p. By the FTC,

$$\int_0^b x^P\,dx = \frac{b^{p+1}}{p+1} - \frac{0}{p+1} = \frac{b^{p+1}}{p+1}$$

as desired. □

This illustrates the power of the FTC to simplify greatly. For instance, as you know from your homework, proving this fact without the FTC takes quite a bit more work. In fact, depending on your approach, it may have been shorter if you just proved the FTC first and then applied it!

The main way that we the second form of the FTC is to deal with integration of the form

$$F(x) = \int_a^{g(x)} f(t)\,dt$$

where $f(x)$ is some continuous and bounded function. Without the FTC, how can we take the derivative? Taking the derivative of an integral itself it subtle without the FTCs, and to deal with the composition with the function $g(x)$ is difficult.

However, we can attack these with the FTC as follows. Let

$$H(x) = \int_a^x f(t)\, dt,$$

so that $F(x) = H(g(x))$. Then the chain rule says

$$F'(x) = H'(g(x)) \cdot g'(x).$$

By the FTC, we see that $H'(x) = f(x)$, so then have

$$F'(x) = f(g(x)) \cdot g'(x),$$

which is something we can calculate!

Let's see this method in action.

Example 2. Calculate the derivative of the function

$$F(x) = \int_0^{x^2} \sin(t) dt.$$

Proof. First, define the function $G(x)$ as

$$G(x) := \int_0^x \sin(t) dt.$$

By the fundamental theorem of calculus, we know that

$$G'(x) := \sin(x).$$

Thus, because $G(x^2) = F(x)$, we can just use the chain rule to see that

$$\begin{aligned}
(F(x))' &= (G(x^2))' \\
&= 2x \cdot G'(x^2) \\
&= 2x \cdot \left(\int_0^x \sin(t) dt \right)' \Bigg|_{x^2} \\
&= 2x \cdot \sin(x^2).
\end{aligned}$$

□

Example 3. Calculate the derivative of the function

$$F(x) = \int_{1/x}^x \frac{1}{t} dt,$$

whenever $t > 0$.

Proof. First, define the function $G(x)$ as

$$G(x) := \int_1^x \frac{1}{t} dt.$$

Then, by the fundamental theorem of calculus, we have that

$$G'(x) := 1/x.$$

So: note that

$$F(x) = \int_{1/x}^{x} \frac{1}{t} dt = \int_{1}^{x} \frac{1}{t} dt - \int_{1}^{1/x} \frac{1}{t} dt = G(x) - G(1/x).$$

(Note that we defined the function G here as an integral starting at 1, not 0! This is because the integral $\int_0^x \frac{1}{t} dt$ doesn't even exist whenever x is nonzero. So, when you use linearity of your integrals to split them apart, do be careful that you're not accidentally breaking your integral into parts that don't exist!)

Then, with this expression of $F(x) = G(x) - G(1/x)$, we can just proceed by the chain rule:

$$\begin{aligned}(F(x))' &= (G(x) - G(1/x))' \\ &= G'(x) - (-\frac{1}{x^2}) \cdot G'(1/x) \\ &= 1/x + \frac{1}{x^2} \cdot \frac{1}{1/x} \\ &= 2/x.\end{aligned}$$

□

3.1. Review of Integration by Parts and Substitution.

Question 4. *What's*

$$\int_1^2 x^2 e^x dx \text{ ?}$$

Proof. Looking at this problem, it doesn't seem like a substitution will be terribly useful: so, let's try to use integration by parts!

How do these kinds of proofs work? Well: what we want to do is look at the quantity we're integrating (in this case, x^2e^x,) and try to divide it into two parts – a "$f(x)$"-part and a "$g'(x)$" part – such that when we apply the relation $\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x)$, our expression gets simpler!

To ensure that our expression does in fact get simpler, we want to select our $f(x)$ and $g'(x)$ such that

(1) we can calculate the derivative $f'(x)$ of $f(x)$ and find a primitive $g(x)$ of $g'(x)$, so that either
(2) the derivative $f'(x)$ of $f(x)$ is **simpler** than the expression $f(x)$, or
(3) the integral $g(x)$ of $g'(x)$ is **simpler** than the expression $g'(x)$.

So: often, this means that you'll want to put quantities like polynomials or $\ln(x)$'s in the $f(x)$ spot, because taking derivatives of these things generally simplifies them. Conversely, things like e^x's or trig functions whose integrals you know are good choices for the integral spot, as they'll not get much more complex and their derivatives are generally no simpler.

Specifically: what should we choose here? Well, the integral of e^x is a particularly easy thing to calculate, as it's just e^x. As well, x^2 becomes much simpler after repeated derivation: consequently, we want to make the choices

$$\begin{aligned} f(x) &= x^2 & g'(x) &= e^x \\ f'(x) &= 2x & g(x) &= e^x, \end{aligned}$$

which then gives us that

$$\begin{aligned}\int_1^2 x^2e^x dx &= f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx\\ &= x^2e^x\Big|_1^2 - \int_1^2 2xe^x dx.\end{aligned}$$

Another integral! Motivated by the same reasons as before, we attack this integral with integration by parts as well, setting

$$\begin{array}{ll} f(x) = 2x & g'(x) = e^x \\ f'(x) = 2 & g(x) = e^x. \end{array}$$

This then tells us that

$$\begin{aligned}\int_1^2 x^2e^x dx &= x^2e^x\Big|_1^2 - \int_1^2 2xe^x dx\\ &= x^2e^x\Big|_1^2 - \left(f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx\right)\\ &= x^2e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - \int_1^2 2e^x dx\right)\\ &= x^2e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - 2e^x\Big|_1^2\right)\\ &= 4e^2 - e^1 - \left(4e^2 - 2e^1 - 2e^2 + 2e^1\right)\\ &= 2e^2 - e^1.\end{aligned}$$

□

Question 5. *What is*

$$\int_0^2 x^2\sin(x^3)dx \ ?$$

Proof. How do we calculate such an integral? Direct methods seem unpromising, and using trig identities seems completely insane. What happens if we try substitution?

Well: our first question is the following: **what should we pick?** This is the only "hard" part about integration by substitution – making the right choice on what to substitute in. In most cases, what you want to do is to find the part of the integral that you don't know how to deal with – i.e. some sort of "obstruction." Then, try to make a substitution that (1) will remove that obstruction, usually such that (2) the derivative of this substitution is somewhere in your formula.

Here, for example, the term $\sin(x^3)$ is definitely an "obstruction" – we haven't developed any techniques for how to directly integrate such things. So, we make a substitution to make this simpler! In specific: Let $g(x) = x^3$. This turns our term $\sin(x^3)$ into a $\sin(g(x))$, which is much easier to deal with Also, the derivative $g'(x) = 3x^2dx$ is (up to a constant) being multiplied by our original formula – so this substitution seems quite promising. In fact, if we calculate and use our indicated substitution, we have that

$$\begin{aligned}\int_0^2 x^2 \sin(x^3)dx &= \int_0^2 \sin(g(x)) \cdot \frac{1}{3} \cdot g'(x)dx \\ &= \int_{0^3}^{2^3} \sin(x)dx \\ &= \frac{\cos(0)}{3} - \frac{\cos(8)}{3}\end{aligned}$$

(Note that when we made our substitution, we also changed the bounds from $[a, b]$ to $[g(a), g(b)]$! Please, please, always change your bounds when you make a substitution!) □

4. Why is differentiation easy, but integration so hard?

As you may have noticed, in calculus, integration is much harder than differentiation. But why is this the case, if they are essentially the same, as inverses of each other?

Well, the first thing is that you shouldn't always expect the existence of inverses operations to have similar levels of difficulty. In other words, "inverse" does not always mean "symmetric." For instance, it's easy to mix a needle into a haystack, but it's harder to then separate the needle from the haystack once again. For a slightly more concrete example, think of squaring and taking the square root of rational numbers. The square of a rational number is always rational, but the square root of a rational number may not be!

Another reason that integration is seen as hard is due to the approach that we tend to take when learning calculus: generally speaking, we learn differentiation first, think of it as the "basic" operation, and then thinking of integration as the inverse.

But what if we tried the opposite point of view? (Indeed, Apostol itself follows this approach by defining integration first, but Math 1a generally takes the traditional "differentiation, then integration" approach.)

For instance, consider the following question. Given a function g, when is there an f such that

$$g(x) = C + \int_0^x f(t)\,dt$$

holds? The answer is whenever g is absolutely continuous, which is when for every $\epsilon > 0$, there exists a δ such that whenever we have a finite sequence of pairwise disjoint subintervals (a_k, b_k) such that

$$\sum_k |b_k - a_k| < \delta$$

then

$$\sum_k |f(b_k) - f(a_k)| < \epsilon.$$

This is stronger than continuity, and stronger than a notion called *uniform* continuity (which itself implies continuity). As you may notice, this definition is difficult to check, unless your function is of a specific type.

Every absolutely continuous function is continuous, but there are lots of continuous functions that are not absolutely continuous. Indeed, "most" continuous are not absolutely continuous, in a sense similar to how we saw that "most" continuous functions are not differentiable.

This actually highlights one instance of why differentiation is easy, but integration is hard. For instance, differentiation just relies on "local" information, while integration relies on "global" information. And it's this difficulty in "patching up" local information to get global information that contributes to integration being more difficult.

5. Taylor Series

5.1. Definitions.

When we first defined the derivative, recall that it was supposed to be the "instantaneous rate of change" of a function $f(x)$ at a given point a. In other words, f' gives us a linear approximation to $f(x)$ near a: for small ε, we have $f(a+\varepsilon) \approx f(a) + \varepsilon f'(a)$.

Taylor series is just the extension of this idea to higher order derivatives, giving us a better approximation to $f(x)$.

Definition 1. Let $f(x)$ be n-times continuously differentiable on an interval $[a, b]$ and let $c \in (a, b)$. Then, the n-th order Taylor polynomial of $f(x)$ about c is:

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

The n-th order remainder of $f(x)$ is defined to be

$$R_n(f)(x) = f(x) - T_n(f)(x)$$

The following two theorems are the essence of why Taylor series is indeed a good approximation of $f(x)$.

Theorem 2. *Let the notation be as in Definition 1. Then, $R_n(f)(x)$ is $o((x-c)^n)$.*

Note that this really just means that when x is close to c, $T_n(f)(x)$ is close to the real value of $f(x)$.

Theorem 3. *Suppose $f(x)$ is $(n+1)$-times continuously differentiable. Then,*

$$R_n(f)(x) = \int_c^x \frac{f^{(n+1)}(c)}{n!}(x-y)^n dy$$

Before using Taylor series to approximate integrals, let's compute the Taylor series for e^x.

Example 4. Let's compute the Taylor series for $f(x) = e^x$ about 0.

For all n, we know that $f^{(n)}(x) = e^x$. Hence, we get

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Note also that the radius of convergence for this series is ∞.

5.2. **Aside.** In this section, we will derive Euler's formula by Taylor series expansion of $e^x, \sin x$ and $\cos x$, which is a very surprising and (in a way) beautiful formula.

The Taylor series for $\sin x$ and $\cos x$ are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

If you look closely, the Taylor series for $\sin x$ and $\cos x$ look like you've taken half of the terms of the Taylor series for $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and alternated signs. So can we find any relation between these functions?

The answer is YES! Let i be the imaginary number, i.e. i is some number (not real) with the property that $i^2 = -1$. As baffling as it might be to raise a number to a *complex* power, let's take a leap of faith and plug in ix in the Taylor series for e^x. We get:

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2} - \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots \\
&= 1 + ix - \frac{x^2}{2} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \cdots\right) \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

So there we have it! $e^{ix} = \cos x + i \sin x$. If you are still not convinced that this formula is surprising, let's plug in $x = \pi$ and see what we get.

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 \Rightarrow e^{i\pi} + 1 = 0$$

The equation we end up with is often said to be the "most beautiful equation in mathematics". $e, \pi, i, 1$ and 0 are some of the most important numbers and to see them appear in one equation is quite surprising indeed!

5.3. **Approximating Integrals.** Now, we will see how Taylor series can help us approximate integrals. For example, consider the Gaussian integral $\int e^{-x^2} dx$. Unfortunately, there is no elementary antiderivative of e^{-x^2}. But using Taylor series, we can approximate the value of this integral.

Example 5. Approximate $\int_0^{1/3} e^{-x^2} dx$ to within 10^{-6} of its actual value.

From now, instead of writing $T_n(e^{-x^2})(x)$ and $R_n(e^{-x^2})(x)$, we will simply write $T_n(x)$ and $R_n(x)$. Then, for any n, we have $e^{-x^2} = T_n(x) + R_n(x)$. Thus,

$$\int_0^{1/3} e^{-x^2} dx = \int_0^{1/3} T_n(x) dx + \int_0^{1/3} R_n(x) dx$$

Note that $T_n(x)$ is just a polynomial. Therefore, $\int_0^{1/3} T_n(x) dx$ is an integral that we can explicitly compute. On the other hand, we know that $R_n(x)$ goes to 0 as n

increases. So the idea is to make $|\int R_n dx|$ small by increasing n: in this case, we want to find n such that $|\int_0^{1/3} R_n(x)dx| < 10^{-6}$.

By Taylor's Theorem, we have

$$R_n(e^{-x})(x) = \int_0^x (-1)^n \frac{e^{-t}}{n!}(x-t)^n dt$$

Note that $R_n(x)$ from above is $R_n(e^{-x^2})(x)$. However, $R_n(e^{-x^2})(x)$ is precisely equal to $R_n(e^{-x})(x^2)$. Hence, we see that

$$R_n(x) = \int_0^{x^2} (-1)^n \frac{e^{-t}}{n!}(x^2-t)^n dt$$

Unfortunately this is not something we can easily integrate. However, we are not interested in the actual value of this integral. We only need to bound it by 10^{-6}.

First, we see that on the interval $[0, x^2]$, e^{-t} is always less than or equal to $e^0 = 1$. And $(x^2-t)^n$ is bounded from above by $(x^2-0)^n = x^{2n}$. Note also that for all $t \in [0, x^2]$, $\frac{e^{-t}}{n!}(x^2-t)^n \geq 0$. Therefore, we get

$$\begin{aligned}
|R_n(x)| &= \left| \int_0^{x^2} (-1)^n \frac{e^{-t}}{n!}(x^2-t)^n dt \right| \\
&= \int_0^{x^2} \frac{e^{-t}}{n!}(x^2-t)^n dt \\
&\leq \int_0^{x^2} \frac{1}{n!} x^{2n} dt \\
&= \frac{x^{2n}}{n!} t \Big|_0^{x^2} \\
&= \frac{x^{2n+2}}{n!}
\end{aligned}$$

In our case, we want $|\int_0^{1/3} R_n(t)dt| < 10^{-6}$. Therefore, we need to find a value of n for which $\frac{1}{n!}\left(\frac{1}{3}\right)^{2n+2} < 10^{-6}$. A little playing around with the inequality will tell you that when we let $n = 3$, the inequality is satisfied.

This means that $\int_0^{1/3} T_3(x)dx$ is within 10^{-6} of the real value of $\int_0^{1/3} e^{-x^2}dx$. Again, $\int_0^{1/3} T_3(x)dx$ is very easy to compute explicitly:

$$\begin{aligned}
\int_0^{\frac{1}{3}} T_3(x)dx &= \int_0^{\frac{1}{3}} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right) \Big|_0^{\frac{1}{3}} \\
&= \frac{147604}{459270}
\end{aligned}$$

This example shows that Taylor series can be used efficiently to approximate integrals. However, we should note that Taylor series works well *near* the point at which we are writing the series. For example, if we were to follow the exact same steps in the above example to approximate $\int_0^2 e^{-x^2}dx$ to within 0.1 of the actual value, we would need to compute $\int_0^2 T_{11}(x)dx$. In our example, we only needed third order Taylor expansion to get an approximate value with error less than 10^{-6}. However, as we get farther away from 0, to get an approximate with similar error bounds, we need to use higher and higher order Taylor polynomials.

One way to fix this is to divide the interval into several subintervals. Then, we can write a Taylor series expansion for $f(x)$ for each interval and approximate the integral over each interval as we did in above example.