MA 8, WEEK 8: COMPLEX NUMBERS

1. Complex Numbers

This is one of the most beautiful stories in mathematics. For many mathematicians, including myself, it is complex analysis that inspired us to become mathematicians, in hopes to develop a theory as elegant, as powerful, and as beautiful as the theory of functions of complex variables. It ties together everything you will have learned in mathematics up to then—analysis in single and many real variables, differential equations, linear and abstract algebra, geometry and topology—and not only develops the theory further by synthesizing these methods, but is able to use reflect the light from these new truths back onto the fields from whence they came. We saw this briefly when we talked about radius of convergence, but of the course, the subject goes much deeper.

We won't be able to fully explore this topic, obviously, but I want to at least show you a glimpse of what this subject holds, to entice or challenge you to pursue mathematics beyond just the required courses.

1.1. **Recap.** Let's quickly recall the facts about complex numbers that we learned in lecture. The complex numbers $\mathbf{C}$ are numbers of the form

$$z = x + yi$$

where x and y are real numbers and i is a square root of -1.

Remark 1. Just an aside for those who've seen complex numbers before. What happens if, instead of choosing $i = \sqrt{-1}$, we choose $i = -\sqrt{-1}$?

The amazing thing about complex numbers is each of these basic operations has a geometric interpretation. We can identify

$$z = x + iy \in \mathbf{C} \leftrightarrow (x, y) \in \mathbf{R}^2$$

and in such a way, every complex number defines a point in the (real) plane $\mathbf{R}^2$ and vice-versa.

We say that the **real part** of z is

$$\mathrm{Re}(z) = x$$

and that the **imaginary part** of z is

$$\mathrm{Im}(z) = y.$$

Of course, these are just the x-component and y-component respectively. If you want to be fancy, you can say that these are "projections of z to the real and imaginary axes, respectively."

We can perform addition, subtraction, multiplication, and division with these numbers, under the well-known relation that $i^2 = -1$. Indeed, one can easily verify the following fact.

Proposition 2. *The complex numbers $\mathbf{C}$ form a field.*

The proof is just a matter of checking the field axioms, and using some familiar properties of the real numbers along the way. However, what's nice about the complex numbers is that in addition to being a field, they have the following special property.

Theorem 3. *(The fundamental theorem of algebra) Any degree $n \geq 1$ polynomial*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \boldsymbol{C}$ has a root in the complex numbers.

A field with this property is said to be **algebraically closed**. Why does it have that name? Recall how we can construct $\mathbf{C}$ by considering $\mathbf{R}$ and adding the root of the polynomial equation

$$x^2 + 1.$$

In other words, we can describe the new number $i = \sqrt{-1}$ by performing algebraic operations like addition/subtraction, multiplication/division, and taking roots. The fundamental theorem of algebra says that once you have the complex numbers, you can't make a bigger field by adjoining elements in this way.

Remark 4. Already, in this setting, we can ask many questions and think about how you might answer them. For instance, what should it mean for a function $f : \mathbf{C} \to \mathbf{C}$ to be continuous? We can think of $\mathbf{C}$ as $\mathbf{R}^2$. Consider the limit definition of continuity. For real numbers, we only had to check convergence from above or below. How many directions do we have to check here?

Of course, we'll find out how to answer these kinds of questions in the rest of Math 1.

The complex numbers also admit the operation of complex conjugation: that is, given $z = x + yi \in \mathbf{C}$, we have its **complex conjugate**

$$\overline{z} = x - yi.$$

Remark 5. Physicists often use z^* to denote complex conjugation. There are certain notational reasons for doing this, involving a mathematical discipline called functional analysis, which forms the foundation for quantum mechanics.

We should view this as a function

$$\bar{\cdot} : \mathbf{C} \to \mathbf{C}$$
$$z \mapsto \overline{z}.$$

It is easy to check that this is a bijection and is its own inverse, because complex conjugation applied twice is just the identity map.

Geometrically, this corresponds to reflection across the x-axis (the real axis). Using this geometry, it's easy to see how basic facts about complex conjugation are true, like how complex conjugation applied twice is the identity and how the a number is fixed under complex conjugation if and only if the number is real.

So we clearly gain a powerful geometric perspective by considering the complex numbers instead of the real numbers. But what have had to sacrifice in order to gain this power? For one, we lose the *ordering* that we have on the real numbers, that is, there is no way to make sense of $\leq$ or $\geq$ for complex numbers.

Proposition 6. *There is no ordering of $\boldsymbol{C}$ that is compatible with the field operations (addition and multiplication).*

Solution. The issue, as you may have guessed, is with i. Suppose that we could put an ordering on the complex numbers and say that $i \geq 0$. However, we then obtain

$$i^2 = -1 \geq 0$$

which is absurd and is not compatible with the usual ordering of the real numbers. What if we set $i \leq 0$? We have $-1 \leq 1$. Suppose that we multiply by i, so

$$-i \geq i.$$

Since $i \leq 0$, we need to flip the inequality. Multiplying by i again, we get

$$-i^2 = -(-1) = 1 \leq i^2 = -1$$

which is absurd. $\square$

Remark 7. This in itself is not a proof of the argument (it requires developing some more definitions), but you can see where the essential problem lies. You get similar issues with trying to compare real numbers in the plane. As an exercise, why won't the *lexicographic ordering* ("dictionary ordering")

$$(a, b) < (c, d)$$

if either $a < c$ or $a = c$ and $b < d$, work to give an ordering on $\mathbf{R}^2$?

Instead of this total ordering, we have to settle for the next best thing. We begin with the elementary observation that for $z = x + yi \in \mathbf{C}$ we have

$$z\overline{z} = x^2 + y^2 \geq 0$$

and is zero if and only if $z = 0$. From this, we define the **magnitude**

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

This also has a geometric interpretation, as the (Euclidean) distance from z to the origin, both viewed as points in $\mathbf{R}^2$. In other words, this is simply the magnitude of the vector (x, y).

From the facts above, you can geometrically see the truth of the following facts. It's also easy to prove them algebraically.

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z}\overline{w}$
- $\overline{z/w} = \overline{z}/\overline{w}$
- $z + \overline{z} = 2\text{Re}(z)$
- $z - \overline{z} = 2i\text{Im}(z)$
- $|z| = |\overline{z}|$

In short, we see that complex conjugation commutes with the field operations (i.e. it doesn't matter whether we conjugate before or after addition) and that it doesn't affect the magnitude of a complex number.

OK, as a sanity check, what is the **reciprocal** of a nonzero complex number $z = x + iy$, that is, the number z^{-1} such that $zz^{-1} = z^{-1}z = 1$?

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

1.2. **The complex exponential function and trigonometric identities.** Another great thing about the complex numbers is that multiplication also corresponds to a geometric operation. We saw this behavior in class, but it may still be a little mysterious as to how it works. Here we will use the tools that we have developed so far to explain why this phenomenon occurs.

Recall that we studied the exponential function

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and showed that it has infinite radius of convergence. This implies that we can extend this definition to the complex plane without worrying about any divergence issues.

We also saw in class that

$$|e^{i\theta}| = 1$$

if θ is real and that we have Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

of which the famous identity

$$e^{i\pi} + 1 = 0$$

is a consequence.

One elementary place where Euler's identity is useful is in proving those trigonometric identities that you're always forgetting. For instance, we have

$$(\cos\theta + i\sin\theta)^2 = e^{i\theta}e^{i\theta} = e^{i2\theta} = \cos 2\theta + i\sin 2\theta.$$

Therefore,

$$\cos^2\theta - \sin^2\theta + i(2\cos\theta\sin\theta) = \cos 2\theta + i\sin 2\theta.$$

By comparing real and imaginary parts, we get the familiar identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\cos\theta\sin\theta.$$

Since this is so easily derived, many mathematicians just remember Euler's identity and rederive the formula whenever they need to get things like triple angle identities.

1.3. **Geometry of multiplication by complex numbers.** From the above properties, we see that we can write any complex number z as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta \in \mathbf{R}$. However, note that θ is not unique, but that one can add any integer multiple of $2\pi i$ to it.

Example 8. For instance,

$$i = 1e^{i\pi/2}$$

and

$$i + 1 = \sqrt{2}e^{i\pi/4}.$$

Remark 9. The non-uniqueness of θ is actually a serious phenomenon that requires some theory to get around. For instance, since the complex logarithm $\log(z)$ of $1 = e^{i0} = e^{i0} \cdot e^{2\pi ik}$ for $k \in \mathbf{Z}$ is

$$\log(1) = \{2\pi ik \mid k \in \mathbf{Z}\}$$

In other words, the complex logarithm is not a function $\log : \mathbf{C} \to \mathbf{C}$ because it maps one value to infinitely many values.

In complex analysis, we deal with this by choosing something called a branch of the logarithm, that is, we simply define the complex logarithm by forcing the θ to take values in some specific region of length 2π, like $[0, 2\pi)$ or $[-\pi, \pi)$. You will explore some of the elementary consequences of this ambiguity in one of your homework problems.

Given two complex numbers $z = r_1 e^{i\theta_1}, w = r_2 e^{i\theta_2} \in \mathbf{C}$, we have

$$zw = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Therefore multiplication by a complex number admits a geometric interpretation; namely, the map

$$\begin{aligned} \phi_w : \mathbf{C} &\to \mathbf{C} \\ z &\mapsto wz \end{aligned}$$

is just a stretching (if $|w| > 1$) or shrinking (if $|w| < 1$) of the complex plane combined with a rotation.

Example 10. For instance, if $w = i$, then ϕ_w is just rotation by 90 degrees.

If $w = i + 1$, then ϕ_w is stretching by $\sqrt{2}$ and rotating by 45 degrees.

Let's quickly investigate how this works for various subsets of $\mathbf{C}$.

1.4. **Geometry of "nice" maps from C to C.** Note that complex conjugation is not a map that can be given by a stretching, rotation, and translation. (There's no way to flip the plane across on axis by any combination of these operations.) This difference is actually crucial and has to do with a property that is satisfied by nice maps like ϕ_w from $\mathbf{C}$ to itself, but is not satisfied by $\bar{\cdot}$. However, the study of this property ("holomorphy") is precisely the subject of complex analysis. They are functions that satisfy a particular differential equation. Among other things, these maps preserve geometric properties: for instance, they preserve all angles; and send lines to either a line or a circles and send circles to either a circle or a lines.

An interesting fact that follows from this theory is that the function $p : \mathbf{C} \to \mathbf{C}$ sending z to any polynomial $p(z)$ is a holomorphic map. In other words, it can also be represented as a composition of a stretching, a rotation, and a translation; in particular it preserves angles and has all the nice properties alluded to above.

For a fun example of these nice maps, consider the map that sends a an element to its inverse:

$$\begin{aligned} i : \mathbf{C} &\to \mathbf{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

Ignore the fact that this map is not strictly defined at 0, since then we would be dividing by zero. We will deal with this issue shortly. This operation (like complex conjugation) is also a map that when applied twice is just the identity. This also admits an awesome geometric interpretation.

For instance, what are the points fixed by this map, that is, what are the $z \in \mathbf{C}$ such that $f(z) = z = \frac{1}{z}$? We see that this amounts to solving

$$z^2 - 1 = (z-1)(z+1)$$

and so ± 1 are the fixed points of this map.

Given a circle of radius r in $\mathbf{C}$, we observe that i maps this circle to a circle of radius $1/r$. In particular, this map fixes the circle of radius 1.

Now, let's make another series of observations. Suppose that we move into three dimensions (where the x- and y-axes are given by the complex plane and z is a real axis) and consider a (hollow) sphere whose "south pole" lies at the origin of the complex plane. Imagine a beam shining from the north pole of the sphere. More precisely, given a point a on the sphere, there is unique line that runs through the north pole and a, and this line intersects the complex plane at one point, called it z_a. Then $a \mapsto z_a$ is a bijective map between points on the complex plane and points on the ball except for the north pole:

$$\mathbf{C} = \text{sphere} - \{\text{north pole}\}.$$

So we now have a separate model for $\mathbf{C}$. We can think of it as a sphere minus a point, in addition to think of it as a $\mathbf{R}^2$ as usual.

Since actual sphere are nicer than spheres with a hole in them, let's add a point called "∞" to the set, so we have

$$\mathbf{C} \cup \{\infty\} = \text{sphere}$$

Consider the point ∞ to be the north pole of this ball. Note that many properties behave well, for instance, we can think of ∞ as a point of infinite distance away from the origin, headed in any direction, and this kind of property is preserved by this construction. For one, with this construction, we can interpret a line on the complex plane as simply a circle on this sphere that goes through the point ∞.

It then turns out that we can naturally extend our function i to

$$i : \mathbf{C} \cup \{\infty\} \mapsto \mathbf{C} \cup \{\infty\}$$
$$z \mapsto \frac{1}{z}.$$

In particular, this map is now well-defined on all points, provided that we map

$$0 \mapsto \text{``}\frac{1}{0}\text{''} = \infty$$
$$\infty \mapsto \text{``}\frac{1}{\infty}\text{''} = 0.$$

Using this sphere model of $\mathbf{C} \cup \{\infty\}$, we can see that i corresponds to the the map that flips this sphere inside out, along the unit circle.

In turns out that all such "nice" functions from $\mathbf{C}$ to itself have this kind of geometric interpretation. In particular, note that from this result, we see that the property I mentioned above that circles must be sent to lines or circles and that lines must be sent to lines or circles, is obvious from this picture. Since our only possible operations are scaling, rotation, and translation, all such maps are just sending circles to other circles on the sphere.