Linked Grassmannian and Local Models of Shimura Varieties

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ABSTRACT. We show that a natural class of local models of Shimura varieties coincide with certain linked Grassmannians, which arise in the theory of limit linear series on algebraic curves. In one direction, this allows us to give a different interpretation of stratifications these local models, in terms of a limit linear series moduli problem. In the other direction, it allows us to completely characterize the geometry of such linked Grassmannians using the group-theoretic machinery developed for studying special fibers of local models; namely, the special fibers of such linked Grassmannians must be unions of Schubert varieties in an appropriate affine flag variety, indexed by a particular subset of the affine Weyl group.

1. INTRODUCTION

We want to describe an isomorphism between certain moduli spaces of limit linear series on algebraic curves and objects that are, over p-adic rings, local models of certain Shimura varieties. Roughly speaking, it implies that the local deformations of certain degenerating line bundles on curves and those of reductions of abelian varieties with additional structure coincide. An overarching reason for this coincidence is currently a mystery to us, especially because these spaces are generally singular. In particular, this connection highlights some group-theoretic structures in the theory of limit linear series and gives a natural alternative moduli-theoretic interpretation for some structures and constructions in the theory of local models.

Our first main result is the following theorem.

Theorem A. *If $M^{\mathrm{loc}} = M^{\mathrm{loc}}(G, \mathcal{L}, \{\mu\})$ is a local model attached to a datum $(G, \mathcal{L}, \{\mu\})$ of unramified type A*[1] *over a discrete valuation ring $\mathcal{O}$ with a choice of uniformizer π, then M^{loc} is isomorphic as an $\mathcal{O}$-scheme to a linked Grassmannian attached to a π-linked chain.*

For the full statement of the theorem, which is more explicit, see Theorem 5.1.

Local models of Shimura varieties (see [PRS13] for a survey) are formulated over the ring of integers of a p-adic field due to their arithmetic origins, but the

[1]Specifically, one associated with a parahoric subgroup and a minuscule cocharacter μ of a restriction of scalars GL_n over an unramified extension of local fields. See §2 for a definition and explanation of the notation.

definition of such a local model extends without modification to a general discrete valuation ring, and our results also hold at this level of generality. Consequently, using results of Osserman [Oss06] and Helm–Osserman [HO08] on linked Grassmannians attached to s-linked chains, we obtain an alternative proof of key geometric properties of local models, such as their flatness, reducedness, normality and Cohen–Macaulayness (proven over p-adic fields in [Gör01] and [He13]) which also holds in our generalized context. In particular, we do not need to assume that the residue characteristic of $\mathcal{O}$ is positive, which is an essential assumption in certain parts of the work of Görtz and He.

By the theory of local models as initiated by Rapoport and Zink [RZ96], an immediate consequence of this theorem is the following result over p-adic rings.

Corollary A. *Let $X = \mathrm{Sh}(G, \{\mu\})$ be a Shimura variety of PEL-type attached to a unitary group that splits over an unramified extension of $\boldsymbol{Q}_p$. Let E denote its reflex field and assume that X has parahoric level structure at a prime $\mathfrak{p}$ of E. Then there exists an isomorphism between étale-local neighborhoods of the special fiber of the associated integral model of X over $\mathcal{O}_{E_\mathfrak{p}}$ and a linked Grassmannian attached to a π-linked chain, where π is a uniformizer of $\mathcal{O}_{E_\mathfrak{p}}$.*

In other words, this shows that the special fiber of an appropriate limit linear series moduli problem also models the singularities of the mod $\mathfrak{p}$ reduction of the integral model. We view the associated linked Grassmannian as a simpler object to study than the local model for several reasons—for one, the formulation of linked Grassmannians do not rely on deep results about deformation spaces of p-divisible groups or specific properties of, say, p-isogenies of abelian varieties—so the fact that they turn out to be identical in these cases and that the linked Grassmannian still models the singularities of the mod $\mathfrak{p}$ reduction of the Shimura variety was initially a little surprising to us.

Also, there are natural points and strata of the linked Grassmannian that have natural interpretations in terms of the limit linear series moduli problem, and in some cases, these correspond precisely to points and strata of the local model that have been studied previously. For example, in the case of the local model associated with the modular curve $X_0(p)$ at p, the non-exact point of the special fiber of the linked Grassmannian attached to the p-linked chain correspond precisely to the point of the special fiber of the local model associated with supersingular points of the mod p reduction of $X_0(p)$ (see §1.1). Furthermore, linked Grassmannians attached to s-linked chains also have a number of favorable properties, including a number of natural stratifications and a simple characterization of the non-smooth points in terms of the limit linear series moduli problem.

Our second main result is in the converse direction.

Theorem B. *Let $\mathcal{O}$ be a discrete valuation ring with a choice of uniformizer π. If $E_\bullet$ is a π-linked chain over $\mathcal{O}$, then its associated linked Grassmannian $\mathrm{LinGr}(r, E_\bullet)$ for any rank r is isomorphic as an $\mathcal{O}$-scheme to an unramified type A local model.*

Once again, the result that we actually prove is more explicit, see Theorem 5.1

For one, this implies that the group-theoretic tools that have been used to analyze the intricate geometry of special fibers of local models of Shimura varieties can be applied to these linked Grassmannians. In particular, the irreducible components of the special fiber are known to be orbits under the action of the corresponding parahoric subgroup, which has some immediate consequences, such as the fact that there exists a common point of intersection for all irreducible components. Furthermore, special fibers of these local models are known to be a union of certain Schubert varieties in an appropriate affine flag variety (e.g. [PR03]), and moreover, we know precisely which Schubert varieties show up, as they are indexed by the μ-admissible/permissible locus of the appropriate affine Weyl group in the Iwahori case (see e.g. [Rap90, §2]). As an application, we can obtain an explicit description of the components of such linked Grassmannians and their intersections, resolving a question of Osserman [Oss06, A.19] in this setting. The relation to Bruhat–Tits trees via the local model perspective can also be viewed as a conceptual explanation as to why there exists an "adaptable basis" for linked Grassmannians attached to s-linked chains (e.g. [EO13, Lem. 2.3], [Oss06, Lem. A.12(ii)]). The connection to local models also indicates that these special fibers of these linked Grassmannians come equipped not only with a torus action, but with an action of a parahoric subgroup containing such a torus.

By combining Theorem A and Theorem B, we obtain the following corollary.

Corollary B. *Over any discrete valuation ring $\mathcal{O}$, an $\mathcal{O}$-scheme is an unramified type A local model of a Shimura variety if and only if it is a linked Grassmannian attached to a π-linked chain.*

While a relation between linked Grassmannians and local models of Shimura varieties seems to have been suspected for some time—for instance, when $\mathcal{O} = \mathbf{Z}_p$, the fact that the linked Grassmannian of rank 1 attached to a p-linked chain of length 2 is isomorphic to the local model of the modular curve $X_0(p)$ is noted in [HO08])—the fact that these objects are one and the same is surprising, at least to the authors. This raises a number of natural questions. In particular, it says that these should be a dictionary between these two types of objects, and we can ask whether notions or constructions that are natural in one setting also admit interpretations in the other setting, or whether different notions on each side (such as natural stratifications of each moduli space) coincide with each other.

Before giving an overview of our overall strategy and beginning our preparations for the proof, let us illustrate our ideas in the simplest case in which the local model and linked Grassmannians are singular. Since there is a paucity of worked

examples of linked Grassmannians, especially a detailed study of their geometry, we give more details about their construction than that of the respective local models. For any unexplained definitions and notation, we refer to the appropriate later sections of the article.

Example 1.1. Consider the (compactified) modular curve $X_0(p)$ with $\Gamma_0(p)$ level structure, so here $G = GL_2(\mathbf{Q}_p)$ and

$$P = \Gamma_0(p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, p \mid c, \ ad \in \mathbf{Z}_p^\times \right\} \subseteq GL_2(\mathbf{Q}_p)$$

is the Iwahori subgroup, and so corresponds to the full standard lattice chain (2.1) for $n = 2$.. The corresponding dominant minuscule cocharacter is $\mu = (1, 0)$ (with respect to the torus of diagonal matrices and Borel subgroup of upper-triangular matrices) and the reflex field is $E = \mathbf{Q}_p$. Thus, the corresponding local model of $X_0(p)$ is defined over $\mathcal{O} = \mathbf{Z}_p$. Let π denote a choice of uniformizer for $\mathcal{O}$.

The local model in this case is obtained by the blowing up the projective line $\mathbf{P}^1_{\mathbf{Z}_p}$ over $\operatorname{Spec} \mathbf{Z}_p$ at the origin 0 of the special fiber $\mathbf{P}^1_{\mathbf{F}_p} = \mathbf{P}^1_{\mathbf{Z}_p} \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \mathbf{F}_p$ (see, e.g. [Hai05, §4.4] for details). This visibly "locally" looks like the well-known Deligne–Rapoport model for $X_0(p)$, whose reduction modulo p consists of two components intersecting transversally at supersingular points [DR73].

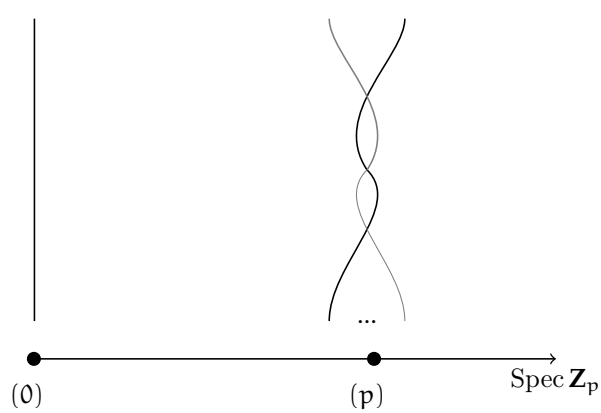

FIGURE 1. The Deligne–Rapoport model of $X_0(p)$

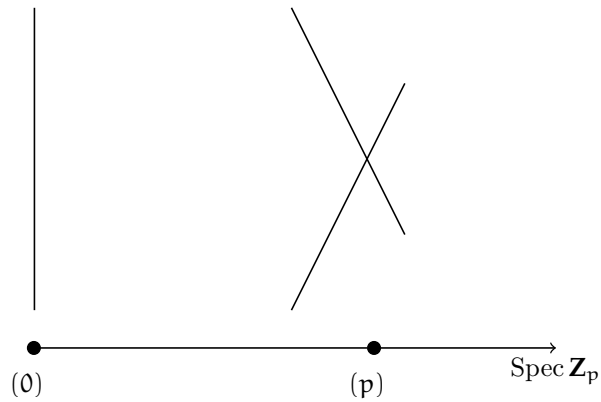

FIGURE 2. The local model for $X_0(p)$, a.k.a. the linked Grassmannian associated to the π-linked chain of length $n = 2$ and rank $r = 1$.

For notational simplicity, we write

$$\mathcal{O}\{a_1, \ldots, a_k\} := \mathrm{span}_{\mathcal{O}}\{a_1, \ldots, a_k\}$$

for the sub-$\mathcal{O}$-module of $\mathrm{Frac}(\mathcal{O})^2$ spanned by vectors $a_1, \ldots, a_k \in \mathrm{Frac}(\mathcal{O})^2$.

Theorem A says that this local model is just the rank-1 linked Grassmannian corresponding to the π-linked chain $E_\bullet$ of length $n = 2$ (cf. 3.1):

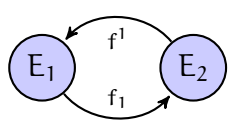

FIGURE 3. The link diagram for a π-linked chain of length $n = 2$.

where

$$E_1 = \mathcal{O}\{e_1, e_2\}, \quad E_2 = \mathcal{O}\{\pi^{-1}e_1, e_2\}$$

and where f_1 and f^1 are the morphisms corresponding to the matrices

$$f_1 = \begin{bmatrix} 1 & \\ & \pi \end{bmatrix} \quad \text{and} \quad f^1 = \begin{bmatrix} \pi & \\ & 1 \end{bmatrix}.$$

The differing geometry of the generic and special fiber of the linked Grassmannian reflect the fact that $\pi = 0$ on the special fiber.

The generic fiber of the corresponding Linked Grassmannian $\mathrm{LinGr}(1, E_\bullet)$ is the classical Grassmannian $\mathrm{Gr}(1,2)_{\mathbf{Q}_p} \cong \mathbf{P}^1_{\mathbf{Q}_p}$. The special fiber consists of two lines that intersect transversely and the point where they intersect corresponds to the unique non-exact point of $\mathrm{LinGr}(1, E_\bullet)$ (cf. 3.2), because the only singularities of linked Grassmannians occur at non-exact points by (Theorem 3.7). Moreover, this is one of the few cases in the literature where there is an explicit (Zariski-)local

description of the geometry of the linked Grassmannian over a fairly general base scheme [HO08, Thm. 3.2].

Proposition 1.2. *The special fiber of* $\mathrm{LinGr}(1, E_\bullet)$ *has a unique non-exact point.*

Proof. Note that on the E_i's, we have

$$\mathrm{Ker}\, f^1 = \mathrm{Im}\, f_1 = \mathcal{O}\{e_1\} \text{ and } \mathrm{Ker}\, f_1 = \mathrm{Im}\, f^1 = \mathcal{O}\{e_2\}.$$

Condition (III) in (3.1) is trivial in this situation, so in order for the exactness condition to fail, we need a linked subbundle (F_1, F_2) such that the restriction of the morphisms f_i and f^i do not intersect the loci above.

The only possible such linked subbundle is the tuple (F_1, F_2) such that

$$F_1 = \mathcal{O}\{e_2\} \text{ and } F_2 = \mathcal{O}\{e_1\}$$

where the restriction of f_1 and f^1 gives

$$\begin{aligned} \mathrm{Ker}\, f^1 &= \mathcal{O}\{e_1\} \\ \mathrm{Im}\, f_1 &= \{0\} \\ \mathrm{Ker}\, f_1 &= \mathcal{O}\{e_2\} \\ \mathrm{Ker}\, f^1 &= \{0\}. \end{aligned}$$

□

Note that a way to deduce the structure of the special fiber of this local model is to observe that the special fiber is the union of the closures of two Iwahori-orbits in the affine flag variety $GL_2(\mathbf{F}_p((t)))/I_{\mathbf{F}_p}$ (where $I_{\mathbf{F}_p}$ is the Iwahori subgroup of $GL_2(\mathbf{F}_p[\![t]\!])$ corresponding to the upper-triangular Borel subgroup), each of which are 1-dimensional and which meet at a point. Exploiting this kind of additional structure is key to formulating the results in a more general setting.

Finally, we note that while the local model and linked Grassmannian turn out to be isomorphic under the identity map given our description above, the isomorphism provided by Theorem between the two spaces is actually *not* an identity map, although it does result in two isomorphic $\mathcal{O}$-schemes. ⋄

The proof of Theorem A and Theorem B proceed in the same way; and indeed, we prove them at exactly the same time by producing an explicit isomorphism in terms of the objects of the respective moduli problems. However, while it is relatively simple to match the data defining one moduli space to another, it is much trickier to explicitly write down the actual map. It turns out that this can be done, to facilitate this process, we need to obtain a definition that generalizes both the local models of Shimura varieties that we study as well as the linked Grassmannians that we hope to relate them to. To do so, we introduce the definition of a *generalized* linked Grassmannian—which we refer to in the paper as simply

"linked Grassmannians," referring to the linked Grassmannians previously studied by Osserman as "classical" linked Grassmannian—and offer an alternative notation and a perspective on the topic, in such a way that our desired isomorphism between $\mathcal{O}$-schemes becomes a consequence of a natural map that exists between linked Grassmannians. For example, it turns out to be very useful to view classical linked Grassmannian not merely as they are usually described, but to define them in terms of periodic lattice chains, inspired by a similar definition given in the theory of local models. Furthermore, when studying both the local model and the classical linked Grassmannian from the perespective of generalized linked Grassmannians, it is natural to introduce an intermediate space—which we dub the the *permadeath* linked Grassmannian—that allows for more obvious maps to and from the local model and the classical linked Grassmannian.

Finally, we conclude our presentation by taking some first steps in applying our knowledge of this isomorphism to obtain interesting results in "both directions": we analyze the geometry of a linked Grassmannian by looking at the additional group-theoretic machinery that we have to analyze the special fiber of corresponding local model, and we obtain a simple "global realization" of a certain class of local models by relating them to some previously studied moduli spaces in the theory of limit linear series.

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2. Local models

We briefly recall the definition of local models, specialized to the context of our theorem. For generalities on the theory of local models, we refer to the survey of Pappas–Rapoport–Smithling [PRS13].

Fix an integer $n \geq 2$. Let $G = \mathrm{GL}_n$, defined over our discrete valuation field F with ring of integers $\mathcal{O}$, and let $\mathcal{L}$ be a **periodic lattice chain in** F^n, that is, a collection of $\mathcal{O}$-lattices in F^n that is (a) totally ordered under inclusion, and (b) if $\Lambda \in \mathcal{L}$, then $a\Lambda \in \mathcal{L}$ for every $a \in F^\times$. The choice of periodic lattice chain in the local model corresponds to the choice of a parahoric subgroup of G obtained as the intersection of the stabilizers of a "period" in $\mathcal{L}$.

Indeed, following the usual convention, we make explicit a periodic lattice chain that is suitable for our purposes. For $i = na + j \in \mathbf{Z}$ with $a \in \mathbf{Z}$ and $0 \leq j < n$, we define the $\mathcal{O}$-lattice

$$\Lambda_i := \sum_{\ell=1}^{j} \pi^{-a-1}\mathcal{O}e_\ell + \sum_{\ell=j+1}^{n} \pi^{-a}\mathcal{O}e_\ell \subset F^n,$$

where $e_1, \ldots, e_n$ denotes the standard ordered basis in F^n. Then the Λ_i's form a periodic lattice chain

$$\cdots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \tag{2.1}$$

called the *standard lattice chain*. Taking our periodic lattice chain $\mathcal{L}$ to be the standard lattice chain corresponds to the choice of the Iwahori subgroup of G (see e.g. [Hai05, §2-3]).

Let $\mathcal{L}$ be a periodic lattice chain in F^n. Fix an integer $0 \leq r \leq n$ and consider the cocharacter $\mu = (1^{(n-r)}, 0^{(r)}) = (1, \ldots, 1, 0, \ldots, 0)$ of the standard maximal torus of diagonal matrices in G. Let $\{\mu\}$ denote the geometric conjugacy class of μ over an algebraic closure $\overline{F}$ of F. We recall the definition of a local model associated with the triple $(G, \{\mu\}, \mathcal{L})$ [PRS13, §2.1].

Definition 2.2. The **local model** $M^{\mathrm{loc}} = M^{\mathrm{loc}}(G, \{\mu\}, \mathcal{L})$ is a functor from the category of $\mathcal{O}_F$-algebras into sets that assigns to each $\mathcal{O}_F$-algebra R the set of all families $\{\mathcal{F}_\Lambda\}_{\Lambda \in \mathcal{L}}$ such that

(i) (*rank*) for every $\Lambda \in \mathcal{L}$, we have an R-submodule $\mathcal{F}_\Lambda$ of $\Lambda \otimes_{\mathcal{O}_F} R$ which is Zariski-locally on $\operatorname{Spec} R$ a direct summand of rank r;

(ii) (*functoriality*) for every inclusion of lattices $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the induced map $\Lambda \otimes_{\mathcal{O}_F} R \to \Lambda' \otimes_{\mathcal{O}_F} R$ carries $\mathcal{F}_\Lambda$ into $\mathcal{F}_{\Lambda'}$:

$$\begin{tikzcd} \Lambda \otimes_{\mathcal{O}_F} R \arrow[r] & \Lambda' \otimes_{\mathcal{O}_F} R \\ \mathcal{F}_\Lambda \arrow[u, hook] \arrow[r, dotted] & \mathcal{F}_{\Lambda'} \arrow[u, hook] \end{tikzcd}$$

(iii) (*periodicity*) for every $a \in F^\times$ and every $\Lambda \in \mathcal{L}$, the isomorphism $\Lambda \xrightarrow{a} a\Lambda$ identifies $\mathcal{F}_\Lambda \xrightarrow{\sim} \mathcal{F}_{a\Lambda}$.

It is immediate from the definition that M^{loc} is a closed subscheme of a product of finitely many copies of $\mathrm{Gr}(r, n)_{\mathcal{O}_F}$, the Grassmannian of r-planes in n-space; namely, $M^{\mathrm{loc}} \subset (\mathrm{Gr}(r, n)_{\mathcal{O}_F})^m$ where m is length of the longest "period" in $\mathcal{L}$ (e.g. for the standard lattice chain (2.1), we have $m = n$). Furthermore, due to the invertibility of the transition maps when we pass to the generic fiber M^{loc}_F, such a family $\{\mathcal{F}_\Lambda\}_{\Lambda \in \mathcal{L}}$ over F is determined by a single choice of $\mathcal{F}_\Lambda$ for any $\Lambda \in \mathcal{L}$, and so we have an isomorphism $M^{\mathrm{loc}}_F \cong \mathrm{Gr}(r, n)_F$.

Finer properties of the geometry of M^{loc} are more subtle to establish. To begin, we have the following results of Görtz for our choice of $(G, \{\mu\}, \mathcal{L})$ as above.

Theorem 2.3. [Gör01, Thm. 4.17–4.21] *The scheme $M^{\mathrm{loc}}(G, \{\mu\}, \mathcal{L})$ is flat over $\operatorname{Spec} \mathcal{O}_F$ with reduced special fiber. The irreducible components of its special fiber are normal with rational singularities (and so are Cohen–Macaulay, in particular).*

Indeed, for our triple $(G, \{\mu\}, \mathcal{L})$, the whole special fiber, not only its irreducible components, are Cohen–Macaulay by a theorem of He [He13], which together

with the generic smoothness of the special fiber given by Görtz's theorem above, implies that M^{loc} is normal by Serre's criterion.

3. Classical Linked Grassmannians

In this section, we recall the definition of ("classical," i.e. finite s-linked chain) linked Grassmannians. We refer the reader to the original papers for details and motivation (cf. [Oss04, Chapter II], [Oss06, Appendix]) and only recall what is necessary for our purposes. Note that, unlike local models, these linked Grassmannians are usually defined over an arbitrary base scheme, and their finer geometric properties also hold under relatively mild assumptions. However, we will not exploit this additional generality in the proof of our main theorem.

Definition 3.1. Let S be a scheme and let $d, m \in \mathbf{Z}_{>0}$. Let $E_1, \ldots, E_m$ be rank d vector bundles on S. Suppose that we have morphisms

$$f_i : E_i \to E_{i+1}, \quad f^i : E_{i+1} \to E_i$$

for all $i = 1, \ldots, m-1$. Given an $s \in \Gamma(S, \mathcal{O}_S)$, we say that the system $E_\bullet = (E_i, f_i, f^i)$ is an s**-linked chain of length** m **and rank** d if the following conditions hold:

(I) For each $i = 1, \ldots, m$,

$$f_i \circ f^i = s \cdot \mathrm{id}_{E_{i+1}}, \text{ and } f^i \circ f_i = s \cdot \mathrm{id}_{E_i}.$$

(II) On the fibers of the E_i at any point with $s = 0$, for every $i = 1, \ldots, m-1$, we have

$$\operatorname{Ker} f^i = \operatorname{Im} f_i, \text{ and } \operatorname{Ker} f_i = \operatorname{Im} f^i.$$

(III) On the fibers of the E_i at any point with $s = 0$, for every $i = 1, \ldots, m-2$, we have

$$\operatorname{Im} f_i \cap \operatorname{Ker} f_{i+1} = (0), \text{ and } \operatorname{Im} f^{i+1} \cap \operatorname{Ker} f^i = (0).$$

We say that the system $E_\bullet$ is **weakly** s-linked if only conditions (I) and (III) hold.

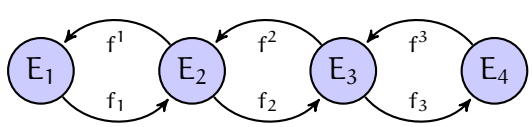

Figure 4. The link diagram for an s-linked chain of length $m = 4$.

Let $E_\bullet$ be weakly s-linked chain of length m and rank d and pick $r < d$. A **linked rank** r **subbundle** $F_\bullet \subseteq E_\bullet$ is an m-tuple of subbundles $F_i \subseteq E_i$ of rank r for $i = 1, \ldots, m$ such that

$$f_i F_i \subseteq F_{i+1}, \quad \text{and} \quad f^i F_{i+1} \subseteq F_i$$

for all $i = 1, \ldots, m-1$.

A linked subbundle is automatically *weakly* s-linked, but it is not generally s-linked.

Definition 3.2. Let S be a scheme. Given an $r \in \mathbf{Z}_{>0}$ and an s-linked chain $E_\bullet$, the **(classical) linked Grassmannian** $\mathrm{LinGr}(r, E_\bullet)$ over S is the closed subscheme of

$$\mathrm{Gr}(r, E_1) \times_S \cdots \times_S \mathrm{Gr}(r, E_m) \tag{3.3}$$

(where $\mathrm{Gr}(r, E_i)$ is the rank r Grassmannian in E_i) consisting of m-tuples $(F_1, \ldots, F_m)$ such that for $i = 1, \ldots, m-1$, we have

$$f_i(F_i) \subseteq F_{i+1}, \text{ and } f^i(F_{i+1}) \subseteq F_i.$$

These linked Grassmannians are known to have nice geometric properties under mild restrictions on the base scheme.

Theorem 3.4. [HO08] *If S is integral and Cohen–Macaulay, then the linked Grassmannian scheme $\mathrm{LinGr}(r, E_\bullet)$ over S is flat (over S), reduced and Cohen–Macaulay, and has reduced fibers.*

The general geometry of the classical linked Grassmannians is a little mysterious if one does not have recourse to the methods like we do in this paper. However, its power is in its abstraction and nonexplicitness and generality. Namely, there are some general structural results that are known. It was these structural similarities that formed the primary motivation for the project that led to this paper.

It is not difficult to see that for any s-linked chain $E_\bullet$ of rank d, the linked Grassmannian $\mathrm{LinGr}(r, E_\bullet)$ exists for all $r < d$. It is a closed subscheme of a product of Grassmannians (3.3), and so is projective. On fibers with $s \neq 0$, condition (I) on s-linked chains implies that f_i and f^i are isomorphisms, so the choice of any subbundle F_i uniquely determines the other subbundles in the chain $F_\bullet = (F_1, \ldots, F_n)$. Thus, the corresponding fiber of $\mathrm{LinGr}(r, E_\bullet)$ is isomorphic to the classical Grassmannian $\mathrm{Gr}(r, d)$.

Thus, we see that much like the local models of §2 the more interesting behavior of the linked Grassmannian occurs on fibers where $s = 0$. An important notion in studying the geometry of such a fiber is the following definition.

Definition 3.5. Given an s-linked chain $E_\bullet$ of length m and rank d on S, a morphism $T \to S$ and a linked subbundle $F_\bullet \subseteq E_\bullet|_T$ of rank r, we say that $F_\bullet$ is an **exact point** of $\mathrm{LinGr}(r, E_\bullet)$ if on the fibers of the F_i at any point of T with $s = 0$, we have

$$\operatorname{Ker} f^i = \operatorname{Im} f_i, \quad \text{and } \operatorname{Ker} f_i = \operatorname{Im} f^i$$

for all $i = 1, \ldots, m-1$.

Equivalently, $F_\bullet$ is an exact point if it is s-linked, and not just weakly s-linked.

The exact points form an open subscheme of $\mathrm{LinGr}(r, E_\bullet)$ that can be explicitly characterized.

Theorem 3.6. [Oss06, A.11] *Let* $(F_1, \ldots, F_m) \in \mathrm{LinGr}(r, E_\bullet)$ *denote the corresponding point in the linked Grassmannian associated with an* s*-linked chain* $E_\bullet$ *of length* m*.*

(i) *The exact points form an open subscheme of* $\mathrm{LinGr}(r, E_\bullet)$*. Namely, they are the complement of the closed subscheme on which*

$$\mathrm{rk}\, f_i|_{F_i} + \mathrm{rk}\, f^i|_{F_{i+1}} < r$$

for some i*.*

(ii) *On fibers over points where where* $s = 0$*, we can describe the exact points as those such that*

$$\mathrm{rk}\, f_i(\mathcal{F}_i) + \mathrm{rk}\, f^i(\mathcal{F}_{i+1}) = r$$

for all i*, even for arbitrary scheme-valued points. Furthermore, an exact point has the property that* $f_i(\mathcal{F}_i)$ *is a subbundle of* $\mathcal{F}_{i+1}$ *and* $f^i(\mathcal{F}_{i+1})$ *is a subbundle of* $\mathcal{F}_i$ *for all* i*.*

The main structural results on the exact points and their relation to the geometry of the linked Grassmannian are encapsulated in the following result.

Theorem 3.7. [Oss06, A.12–13] *Let* $E_\bullet$ *be an* s*-linked chain of rank* d*. Pick an integer* $r < d$*. The exact points of* $\mathrm{LinGr}(r, E_\bullet)$ *are precisely the smooth points of* $\mathrm{LinGr}(r, E_\bullet)$ *over* S *and are dense in* $\mathrm{LinGr}(r, E_\bullet)$*. Furthermore, the exact points are dense in every fiber of* $\mathrm{LinGr}(r, E_\bullet)$*.*

Indeed, we can characterize the non-exact points of a fiber to be precisely the intersections of the components of that fiber.

Using the smoothness at the exact points, it is not hard to deduce that the every component of a fiber of any classical linked Grassmannian must be $r(d-r)$-dimensional [Oss06, A.14].

4. Generalized Linked Grassmannians

A primary principle behind the proof of our main theorem (Theorem 5.1) is to work with an abstraction that allows for the uniform treatment of both the local models of Shimura varieties (§2) and "classical" linked Grassmannians (§3); which we call *generalized* linked Grassmannians. From this perspective, a generalized linked Grassmannian can be thought of as a certain flat degeneration of a classical Grassmannian associated with a bound quiver (a.k.a. quiver with relations) and subrepresentations of a fixed representation of such a quiver, or equivalently, as a flat deformation of a quiver Grassmannian (attached to a bound quiver) with uniform dimension vector. In subsequent sections, when we refer to linked Grassmannians, we mean in this generalized sense; when we want to refer to the linked Grassmannians of §3, we will refer to them as *classical* linked Grassmannians.

So far, the machinery of (generalized) linked Grassmannians has only been developed for s-linked chains (cf. 3.1), i.e. classical linked Grassmannians, and all general results on linked Grassmannians currently available in the literature apply only in this case. However, to highlight the versatility of this framework, we develop the rudiments of a general theory of linked Grassmannians and use it both as a conceptual framework and as the fundamental objects that we use in our proof.

4.1. **Defining Generalized Linked Grassmannians.** Let S be a scheme. Generalized linked Grassmannians are attached to objects called link graphs, which are essentially quivers equipped with a choice of quiver representation with uniform dimension vector $(k, k, \ldots, k)$ and which can be chosen to satisfy certain compatibility conditions or relations.

Definition 4.1. Let Γ be a finite directed graph with vertex set Γ_0 and edge set Γ_1, and let $d \geq 1$ be an integer. Suppose that

- for every vertex $v \in \Gamma_0$, we are given a vector bundle E_v of fixed rank d (on S);
- for every edge $e = (v_1 \to v_2) \in \Gamma_1$, we have a map $f_e : E_{v_1} \to E_{v_2}$ (defined over S); and
- we impose certain **link conditions** that put certain restrictions on the maps, such as their behavior under composition or on how their kernels or images relate to each other.

We refer to this data as the **link graph** $\Gamma_\bullet = (\Gamma, d, E_\bullet, f_\bullet)$.

Definition 4.2. Let $r \geq 1$ be an integer such that $r < d$. A **(generalized) linked Grassmannian** attached to a link graph $\Gamma_\bullet$ is a functor $\mathrm{LinGr}(r, \Gamma_\bullet)$ from S-schemes T to the set of $\#\Gamma_0$-tuples $(W_v)_{v \in \Gamma_0}$ of rank r subbundles, where

- $W_v \subseteq E_v$ for each $v \in \Gamma_0$, and
- for every $e = (v_1 \to v_2) \in \Gamma_1$, we have $f_e(W_{v_1}) \subseteq W_{v_2}$.

Example 4.3. If S is affine and there are no link conditions, this is just a quiver Grassmannian where all the submodules have the same rank r.

It is not hard to prove the following basic result.

Proposition 4.4. *The functor* $\mathrm{LinGr}(r, \Gamma_\bullet)$ *is represented by a projective scheme over* S*, which is a closed subscheme of a product of* $\#\Gamma_0$ *Grassmannian schemes over* S*.*

Both the local models (§2) and classical linked Grassmannians (§3) are examples of linked Grassmannians, the only difference coming in their link graphs, which are s-cyclic link graphs and s-linked chains, respectively. As the latter is obviously a linked Grassmannian, we only need to describe the local model in the language of linked Grassmannians.

4.2. s**-Cyclic Linked Grassmannians.** We define s-cyclic link graphs and define the corresponding linked Grassmannian, which we call the s-cyclic linked Grassmannian. The "unramified type A" local models of §2 are special cases of these linked Grassmannians.

We first describe the link graph. Pick an integer $m \geq 1$. Let Γ be a directed graph with m vertices $\Gamma_0 = \{v_1, \ldots, v_m\}$, whose edge set is

$$\Gamma_1 = \{(v_1 \to v_2), (v_2 \to v_3), \ldots, (v_{m-1} \to v_m), (v_m \to v_1)\}.$$

Fix an integer $d \geq 1$ and let C_i denote the rank d vector bundle associated with the vertex v_i. Choose a global section $s \in \Gamma(S, \mathcal{O}_S)$. We impose the following link condition (Def. 4.1): for each $i \in \{1, \ldots, m\}$, the map $C_i \to C_i$ obtained by composing the maps coming from a cycle of edges in Γ is multiplication by s; namely, if we write the edges as $e_i = (v_i \to v_{i+1}) \in \Gamma_1$ (where we take v_{m+1} to be v_1) and f_{e_i} for the corresponding map, then

$$f_{e_{i-1}} \circ \cdots \circ f_{e_{i+1}} \circ f_{e_i} = s \cdot \mathrm{id}_{E_i}$$

for all i. This data $C_\bullet = (\Gamma, d, \{C_i\}_{i=1}^m, \{f_i\}_{i=1}^m)$ with the corresponding link condition is what we call an s**-cyclic link graph** $\Gamma_\bullet$ **of length** m.

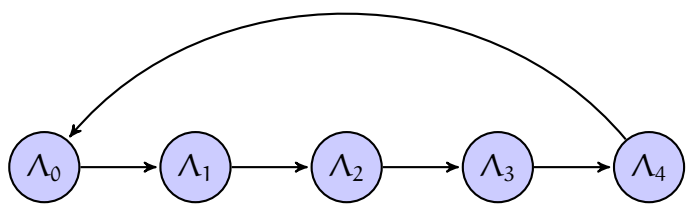

FIGURE 5. The cyclic link graph of length 5 associated with the A_4 local model with Iwahori level structure.

Definition 4.5. Let $r \geq 1$ be an integer such that $r < m$. An s**-cyclic linked Grassmannian** $\mathrm{LinGr}(r, C_\bullet)$ is the linked Grassmannian attached to an s-cyclic link graph $\Gamma_\bullet$ of length m.

Proposition 4.6. *Let $\mathcal{O}$ be a discrete valuation ring and choose a uniformizer π of $\mathcal{O}$. The local model over $\mathcal{O}$ attached to the triple $(\mathrm{GL}_n, \{\mu\}, \mathcal{L})$ is a π-cyclic linked Grassmannian $\mathrm{LinGr}(r, C_\bullet)$.*

Proof. This is just rephrasing the definition of the local model given in (§2) into the language of linked Grassmannians by extracting the essential data from a single period of the periodic lattice chain. Namely, the inputs into the local model are completely determined by the following information:

- a positive integer n, corresponding to the group GL_n;

- a subset $I = \{i_0, \ldots, i_{m-1}\} \subset [n]$ corresponding to the parahoric subgroup P the stabilizes of the lattices Λ_{i_k} in a period fo the periodic lattice chain $\mathcal{L}$; and
- an integer $r \in [n]$ that corresponds to the choice of minuscule cocharacter μ of GL_n.

The associated cyclic linked Grassmannian is $\mathrm{CycLinGr}(r, \Lambda_\bullet)$ where $\Lambda_\bullet$ is the cyclic link chain of length $\#I$ with ambient bundle of rank n, where the bundles correspond to Λ_{i_k} for $i_k \in I$. The morphisms f_i are determined by the induced maps on $\mathcal{L}$. $\square$

4.3. **Defining linked Grassmannians in terms of Periodic Lattice Chains.** For our proofs, we have found it useful to view the linked Grassmannians we consider as not just that of a single link diagram, but to extend the link diagram so as to obtain a periodic lattice chain, to obtain something like definition 2.2 for the linked Grassmannians we consider. This ends up simplifying a number of technical aspects of the proofs, especially for the linked Grassmannians that are *not* also local models attached to an unramified type A datum.

For π-cyclic linked Grassmannians this is immediate from the construction we give in §2. To do this for the the classical linked Grassmannians, we simply allow ourselves to continue following the morphisms f_i and f^i of the π-linked chains.

For concreteness, we give a definition of the classical linked Grassmannian in such a form. Let $\mathcal{E}$ denote the periodic lattice chain consisting of all the lattices of the form aE_i for all $a \in F^\times$ and all E_i in the π-linked chain $E_\bullet$.

Definition 4.7. (Alternative Definition) The **classical linked Grassmannian** $\mathrm{LinGr}(r, E_\bullet)$ is a functor from the category of $\mathcal{O}_F$-algebras into sets that assigns to each $\mathcal{O}_F$-algebra R the set of all families $\{\mathcal{F}_E\}_{E \in \mathcal{E}}$ such that

(i) (*rank* for every $E \in \mathcal{E}$, we have an R-submodule $\mathcal{F}_E$ of $E \otimes_{\mathcal{O}_F} R$ which is Zariski-locally on $\mathrm{Spec}\, R$ a direct summand of rank r;

(ii) (*functoriality*) for every inclusion of lattices $E \subset E'$ in $\mathcal{E}$, the induced map $E \otimes_{\mathcal{O}_F} R \to E' \otimes_{\mathcal{O}_F} R$ carries $\mathcal{F}_E$ into $\mathcal{F}_{E'}$:

$$\begin{tikzcd} E \otimes_{\mathcal{O}_F} R \arrow[r] & E' \otimes_{\mathcal{O}_F} R \\ \mathcal{F}_E \arrow[u, hook] \arrow[r, dotted] & \mathcal{F}_{E'} \arrow[u, hook] \end{tikzcd}$$

(iii) (*periodicity*) for every $a \in F^\times$ and every $E \in \mathcal{L}$, the isomorphism $E \xrightarrow{a} aE$ identifies $\mathcal{F}_E \xrightarrow{\sim} \mathcal{F}_{aE}$.

This looks nearly identical to Definition 2.2, but the property of functoriality has a much stronger consequence for the periodic lattice chain $\mathcal{E}$ coming from the π-linked chain than from that of $\mathcal{L}$ associated with the π-cyclic link diagram. This

is because the π-linked chain has *many* ways to "multiply by π"—as every cycle between two adjacent vertices gives such a way (and technically speaking, it gives two ways, as we traverse it in both directions)—whereas the π-cyclic linked chain has only one way; namely, by following all the maps until you return to the vertex you started with. In particular, for π-linked chains, the multiple of π that you see upon returning to your starting vertex indicates the number of cycles in the link diagram that you completed in your path. For example, for $\mathcal{E}$, we have not just one, but an entire collection of inclusions between our choices of lattices. For example, for every pair $i < j$ of induces in $E_\bullet$, we have the inclusion

$$\cdots \subset \pi^{j-i}E_j \subset E_i \subset E_j \subset \pi^{i-j}E_i \subset \pi^{i-j}E_j \subset \cdots$$

where the inclusion $E_i \hookrightarrow \pi^{i-j}E_i$ is the one coming from the composition

$$f^i \circ f^{i+1} \cdots \circ f^{j-2} \circ f^{j-1} \circ f_{j-1} \circ \cdots f_{i+1} \circ f_i$$

in $E_\bullet$; and there a completely analogous composition corresponding to the inclusion $E_j \hookrightarrow \pi^{i-j}E_j$. However, in addition to these "obvious" inclusions, we get similar inclusions coming from composing the maps above in the opposite order. Furthermore, there is also the option of not taking the "most direct" route back to your starting vertex in inclusions above, as you can also decide to, say, loop around some middle vertices a number of times or go back and forth along the link diagram before returning. Finally, there is nothing special about taking just two indices here, and this kind of nested inclusion diagram also holds for every single collection of indices $I \subset [n]$.

Despite these additional intricacies coming from the property of functoriality, it is not difficult to see that this agrees with Definition 3.2, as they parameterize the exact same moduli problem and are determined by the same data. Furthermore, this kind of periodic lattice chain definition can be generalized to any link diagram that contains cycles that correspond to a quiver relation given by multiplication.

This periodic lattice chain perspective on the local model and classical linked Grassmannian is an immediate consequence of embedding the π-linked chain into the Bruhat–Tits building associated with $GL_n(F)$ in such a way that respects the maps defining the quiver.

Notation 4.8. In our proofs, we use the notation $\sim$ to denote moving up or down a period in the lattice chain.

5. The proof of Theorems A and B

Theorem 5.1. *Let $\mathcal{O}$ be a discrete valuation ring and fix a choice of uniformizer π.*

Let $G = GL_n$ and pick a minuscule cocharacter $\mu = (1^{(n-r)}, 0^{(r)})$. Write $\mathcal{L}$ for an appropriate subset of the full periodic standard lattice chain, corresponding to the choice of parahoric subgroup of G, and let $I \subset [n] = \{1, 2, \ldots, n\}$ denote the corresponding subset of one "period" corresponding to such a choice.

Let $E_\bullet$ be a π-linked chain of length $\#I$ and rank n, where the morphisms f_i and f^i of $E_\bullet$ correspond to the relative positions of a the subset of a period of the periodic lattice chain above.

Then we have an isomorphism of $\mathcal{O}$-schemes $M^{\mathrm{loc}}(G, \{\mu\}, \mathcal{L}) \cong \mathrm{LinGr}(r, E_\bullet)$.

Roughly speaking, the proof can be described by showing that there are two "evident" isomorphisms between *three* (generalized) linked Grassmannians: a π-cyclic linked Grassmannian $\mathrm{LinGr}(r, C_\bullet) = M^{\mathrm{loc}}$ (a.k.a. a local model whose associated group is GL_n), a permadeath linked Grassmannian $\mathrm{LinGr}(r, D_\bullet)$, and the classical linked Grassmannian $\mathrm{LinGr}(r, E_\bullet)$:

$$\mathrm{LinGr}(r, C_\bullet) \xrightarrow[\sim]{\text{Thm. 5.3}} \mathrm{LinGr}(r, D_\bullet) \xrightarrow[\sim]{\text{Thm. 5.9}} \mathrm{LinGr}(r, E_\bullet)$$

The permadeath linked Grassmannian $\mathrm{LinGr}(r, D_\bullet)$ does not seem to have been previously studied (we define it in §5.2), but it is a natural intermediate moduli space for passing data between the local model and classical linked Grassmannian.

Geometrically, these isomorphisms can be viewed as "unfolding an $(n-1)$-simplex" in the Bruhat–Tits building attached to GL_n over F and applying an involution to the building that fixes our choice of origin vertex. In particular, it indicates that the isomorphism of cyclic linked Grassmannian and classical linked Grassmannian are but two of a family of similarly defined isomorphic $\mathcal{O}$-schemes; the others are linked Grassmannians attached to different configurations of link diagrams that can also be viewed as lying in the Bruhat–Tits building.

5.1. **Notation and Conventions.** Fix an integer $n \geq 2$. For any $i \in [n] = \{1, 2, \ldots, n\}$, we let $\Pi(i)$ denote the $n \times n$ diagonal matrix with a π in the ith place and 1's at all other places on the diagonal. More generally, for any subset $I \subset [n]$, we write

$$\Pi(I) := \prod_{i \in I} \Pi(i)$$

for the $n \times n$ diagonal matrix with a π in the ith place for all $i \in I$ and 1's at all other places on the diagonal.

For notational simplicity, we omit the brackets for the sets $I \subset [n]$ when we explicitly list the elements of I. For example, we write $\Pi(i_1, i_2, i_3) := \Pi(\{i_1, i_2, i_3\})$ for any subset $\{i_1, i_2, i_3\} \subset [n]$.

Any element Λ_i in the standard lattice chain (2.1) can be identified with $\mathcal{O}^n$, in which case the diagram

$$\Lambda_0 \hookrightarrow \Lambda_1 \hookrightarrow \Lambda_2 \hookrightarrow \cdots \hookrightarrow \Lambda_{n-1} \hookrightarrow \pi^{-1}\Lambda_0$$

becomes

$$\mathcal{O}^n \xrightarrow{\Pi(1)} \mathcal{O}^n \xrightarrow{\Pi(2)} \cdots \xrightarrow{\Pi(n-1)} \mathcal{O}^n \xrightarrow{\Pi(n)} \mathcal{O}^n,$$

and we work with local models under this identification, suitably extended to the full infinite periodic lattice chain.

We prove our results *in the Iwahori setting*, that is, where the corresponding local model has Iwahori level structure; this means that all the linked Grassmannians here parameterize n-tuples (as opposed to k-tuples for some $k < n$) of bundles of a chosen rank r. The cases corresponding to local models attached to a general parahoric group follow from easy modifications of the proof in the Iwahori case. Namely, if a general parahoric group corresponds to a stabilizer of the subset corresponding to $J \subset [n]$ of the standard $\mathcal{O}$-lattice chain, we consider the $\#J$-tuples corresponding to the elements at the jth position of the n-tuple for all $j \in J$.

Finally, since the generalized linked Grassmannians are moduli spaces that represent the corresponding functors from $\mathcal{O}$-schemes to sets, we can think of the objects in our moduli spaces as (a family of) modules over an $\mathcal{O}$-scheme T. Our arguments are taken with this perspective.

5.2. **Introducing: The Permadeath Linked Grassmannian.**

Definition 5.2. Given an $r \in [n]$, the **rank** r **permadeath linked Grassmannian of length** n is the linked Grassmannian $\mathrm{LinGr}(r, D_\bullet)$ attached to the link diagram $D_\bullet$

$$\begin{tikzcd} D_1 \arrow[r, bend left, "g_1"] & D_2 \arrow[l, bend left, "g^1"] \arrow[r, bend left, "g_2"] & \cdots \arrow[l, bend left, "g^2"] \arrow[r, bend left, "g_{n-2}"] & D_{n-1} \arrow[l, bend left, "g^{n-2}"] \arrow[r, bend left, "g_{n-1}"] & D_n \arrow[l, bend left, "g^{n-1}"] \end{tikzcd}$$

where each D_i is a vector bundle of rank n and the morphisms are defined to be

$$g_i = \Pi(1, 2, \ldots, i) = \prod_{j=1}^{i} \Pi(j) \quad \text{and} \quad g^i = \Pi(i+1, i+2, \ldots, n) = \prod_{j=i+1}^{n} \Pi(j).$$

Remark 1. The name "permadeath" (short for "permanent death") comes from property that if a vector is "killed" by some g_i (i.e. lies in the kernel of g_i) on the special fiber, it must "stay dead" for any g_j with $j > i$ (i.e. lie in the kernel of such a g_j); similarly, if a vector is "killed" by some g^i, then it must "stay dead" for any g^j with $j < i$.

The permadeath linked Grassmannian is close to, but is not quite, the classical linked Grassmannian attached to a π-linked chain. In particular, condition (III) of Definition 3.1 may fail. However, it is immediate that is nonetheless a linked Grassmannian in the sense of §4. We do not delve into a finer study of $\mathrm{LinGr}(r, D_\bullet)$ or determine its more subtle geometric properties, as they are not necessary for our proof.

5.3. **Cyclic to Permadeath.**

Theorem 5.3. *There is an isomorphism of $\mathcal{O}$-schemes induced by the map on objects in the moduli spaces*

$$\begin{aligned}\mathrm{LinGr}(r, C_\bullet) &\to \mathrm{LinGr}(r, D_\bullet)\\ (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1}) &\mapsto \Bigg(\mathcal{F}_0, \mathcal{F}_1, \Pi(1)\mathcal{F}_2, \Pi(1)\Pi(1,2)\mathcal{F}_3,\\ &\qquad \dots, \left(\prod_{j=1}^{n-2} \Pi(1,2,\dots,j)\right)\mathcal{F}_{n-1}\Bigg).\end{aligned}$$

Proof. From the definition of the link diagram $C_\bullet$, we have

$$\Pi(i)\mathcal{F}_{i-1} \subseteq \mathcal{F}_i \tag{5.4}$$

for $i = 1, \dots, n-1$ and $\Pi(n)\mathcal{F}_{n-1} \subseteq \mathcal{F}_0$. Applying this relation successively, we obtain the relation

$$\Pi(i+1, i+2, \dots, n)\Pi(1, 2, \dots, i-1)\mathcal{F}_i \subseteq \mathcal{F}_{i-1}. \tag{5.5}$$

As we have an obvious inverse map via the periodicity property of local models (Definition 2.2(iii)) with respect to multiplication by a power of π, the only nontrivial part of the theorem is to show that the map is well-defined. Namely, we need to show that

$$g_i\left(\left(\prod_{j=1}^{i-2} \Pi(1,2,\dots,j)\right)\mathcal{F}_{i-1}\right) \subseteq \left(\prod_{j=1}^{i-1} \Pi(1,2,\dots,j)\right)\mathcal{F}_i \tag{5.6}$$

and

$$g^i\left(\left(\prod_{j=1}^{i-1} \Pi(1,2,\dots,j)\right)\mathcal{F}_i\right) \subseteq \left(\prod_{j=1}^{i-2} \Pi(1,2,\dots,j)\right)\mathcal{F}_{i-1} \tag{5.7}$$

for all $i = 1, \dots, n-1$.

For the former class of relations, we note that

$$\begin{aligned}g_i(\Pi(1,2,\dots,i-2)\mathcal{F}_{i-1}) &= \Pi(1,2,\dots,i)\Pi(1,2,\dots,i-2)\mathcal{F}_{i-1}\\ &= \Pi(1,2,\dots,i-1)\Pi(1,2,\dots,i-2)\Pi(i)\mathcal{F}_{i-1}\\ &\subseteq \Pi(1,2,\dots,i-1)\Pi(1,2,\dots,i-2)\mathcal{F}_i \quad [\text{by } (5.4)]\\ &\subseteq \Pi(1,2,\dots,i-1)\mathcal{F}_i,\end{aligned}$$

which implies the relation (5.6).

For the latter class of relations, we note that

$$\begin{aligned}g^i(\Pi(1,2,\dots,i-1)\mathcal{F}_i) &= \Pi(i+1, i+2, \dots, n)\Pi(1,2,\dots,i-1)\mathcal{F}_i\\ &\subseteq \mathcal{F}_{i-1} \quad [\text{by } (5.5)],\end{aligned}$$

which implies the relation (5.7). □

The above proof is very slick but is also quite opaque, hiding almost all of the intuition behind the existence of the map in the first place and why the permadeath linked Grassmannian is a natural object to consider in its own right. We include the following alternative proof of the statement above, as it highlights the more geometric aspects of the picture. It is not as self-contained and we have to use the both the known geometric properties and the isomorphism we prove in the following section to conclude our result, but it illustrates a little better what exactly is going on under this map and why this is actually a natural intermediate step and so is simpler than trying to pass to the classical linked Grassmannian directly.

Alternative Proof of Theorem 5.3. Let $(F_1, F_2, \ldots, F_m)$ be an m-tuple of cyclic linked subbundles, that is, a point in $\mathrm{LinGr}(r, C_\bullet)$. (Note that this indexing convention is not the same as the one used in the above proof.) Each F_i can be represented as a lattice generated by r vectors

$$\mathcal{L}_i := \mathcal{O}\{v_1^i, \ldots, v_r^i\}$$

where a generating vector is of the form

$$v_j^i = a_j^i(1)e_1 + a_i^j(2)e_2 + \cdots + a_i^j(m)e_m =: \mathcal{O}([a_j^i(1), \ldots, a_j^i(m)]) \in \mathcal{O}^m,$$

and the cyclic linked condition imposes some compatibility conditions on the coefficients $a_j^i(k)$.

Consider the m-tuple of subbundles $(F_1', F_2', \ldots, F_m')$, where F_i' corresponds to a rank r lattice $\mathcal{L}_i'$ where

$$\mathcal{L}_i' = \mathcal{O}\{w_1^i, \ldots, w_r^i\}$$

where each generating vector is of the form

$$w_j^i = \mathcal{O}([\pi^{i-2}a_j^i(1), \pi^{i-3}a_j^i(2), \ldots, \pi a_j^i(i-2), a_j^i(i-1), a_j^i(i) \ldots, a_j^i(m)]). \tag{5.8}$$

These subbundles $(F_1', F_2', \ldots, F_m')$ define a point in $\mathrm{LinGr}(r, D_\bullet)$, because the subbundles inherit the linked structure of the cyclic link graph and the insertion of the powers of π's in (5.8) ensures that the "permadeath property" holds.

We now want to show that the rule $(F_1, \ldots, F_m) \mapsto (F_1', \ldots, F_m')$ gives us an injective map

$$\mathrm{LinGr}(r, C_\bullet) \hookrightarrow \mathrm{LinGr}(r, D_\bullet).$$

On $\mathcal{O}$-points of the respective functors, we see that the coefficients as defined above are uniquely determined by their behavior on the special fiber and the generic fiber. If two tuples of subbundles in $\mathrm{LinGr}(r, C_\bullet)$ map to the same bundle of $\mathrm{LinGr}(r, D_\bullet)$, the coefficients above must be identical. This is because they are, of course, isomorphic on the special fiber, where $\pi = 0$, but we can detect the power of π of the realization (5.8) by looking at the generic fiber.

We have the commutative diagram

$$\begin{tikzcd}
\mathrm{LinGr}(r, C_\bullet) \arrow[r] \arrow[d] & \mathrm{LinGr}(r, D_\bullet) \arrow[d] \\
\prod_{i=1}^{m} \mathrm{Gr}(r, \Lambda_i) \arrow[r] & \mathrm{Gr}(r, E_1) \times \mathrm{Gr}(r, E_2) \times \prod_{i=3}^{m} \mathrm{Gr}(r, E_i') =: \mathcal{G}.
\end{tikzcd}$$

Since $\mathrm{LinGr}(r, C_\bullet)$ and $\mathrm{LinGr}(r, D_\bullet) \cong \mathrm{LinGr}(m-r, E_\bullet)$ are both flat over the spectrum of the discrete valuation ring $\mathcal{O}$ (by [Gör01] and [HO08] respectively) and their generic fibers are *equal* (not just isomorphic!) as subschemes of $\mathcal{G}$, by the valuative criterion for flatness [Gro67, EGA IV.3, 3.11.8], the two $\mathcal{O}$-schemes are isomorphic as subschemes of the product of Grassmannians $\mathcal{G}$. □

5.4. **Permadeath to Classical.**

Theorem 5.9. *There is an isomorphism of $\mathcal{O}$-schemes induced by the map on objects in the moduli spaces*

$$\begin{aligned}
\mathrm{LinGr}(r, D_\bullet) &\to \mathrm{LinGr}(r, E_\bullet) \\
(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n) &\mapsto \Bigg(\mathcal{G}_1, \Pi(2, \ldots, n)^2 \mathcal{G}_2, \Pi(2, \ldots, n)^2 \Pi(3, \ldots, n)^2 \mathcal{G}_3, \\
&\qquad \ldots, \left(\prod_{j=1}^{n-1} \Pi(j+1, j+2, \ldots, n)^2 \right) \mathcal{G}_n \Bigg)
\end{aligned}$$

Notation 5.10. We have changed the indexing convention here to start from 1 instead of 0; this correspond better to the maps f_i and f^i of the linked diagram for the classical linked Grassmannian of length n. For cyclic linked Grassmannians, we started our indices from 0 so as to better agree with the conventions in the literature for local models; namely, so that the indices of the subbundles agrees with the indices of the elements in the standard lattice chain (2.1) that correspond to the ambient space for such a subbundle.

Proof. The argument is similar in spirit to that of Theorem 5.3. By the definition of the morphisms in the permadeath link diagram $D_\bullet$, we have

$$g_i(\mathcal{G}_i) = \Pi(1, 2, \ldots, i)\mathcal{G}_i \subseteq \mathcal{G}_{i+1} \tag{5.11}$$

and

$$g^i(\mathcal{G}_{i+1}) = \Pi(i+1, i+2, \ldots, n)\mathcal{G}_{i+1} \subseteq \mathcal{G}_i. \tag{5.12}$$

Again, we have an obvious inverse map by using the periodicity property of generalized linked Grassmannians. Thus, it only remains to show that the map is well defined, by showing that

$$f_i \left(\left(\prod_{j=1}^{i-1} \Pi(j+1, j+2, \ldots, n)^2 \right) \mathcal{G}_i \right) \subseteq \left(\prod_{j=1}^{i} \Pi(j+1, j+2, \ldots, n)^2 \right) \mathcal{G}_{i+1} \tag{5.13}$$

and

(5.14)
$$f^i\left(\left(\prod_{j=1}^{i}\Pi(j+1,j+2,\dots,n)^2\right)\mathcal{G}_{i+1}\right)\subseteq\left(\prod_{j=1}^{i-1}\Pi(j+1,j+2,\dots,n)^2\right)\mathcal{G}_i$$

for all $i=1,2,\dots,n-1$.

For the former class of relations, we note that

$$\begin{aligned}f_i\left(\mathcal{G}_i\right)&=g^i\left(\mathcal{G}_i\right)\\&=\Pi(i+1,i+2,\dots,n)\mathcal{G}_i\\&\sim\Pi(1,2,\dots,i)\Pi(i+1,\dots,n)^2\mathcal{G}_i\\&\subseteq\Pi(i+1,\dots,n)^2\mathcal{G}_{i+1}\quad[\text{by }(5.11)],\end{aligned}$$

which implies the relation (5.13).

For the latter class of relations, we note that

$$\begin{aligned}f^i\left(\Pi(i+1,i+2,\dots,n)^2\mathcal{G}_{i+1}\right)&=g_i(\Pi(i+1,i+2,\dots,n)^2\mathcal{G}_{i+1})\\&=\Pi(1,2,\dots,i)\Pi(i+1,i+2,\dots,n)^2\mathcal{G}_{i+1}\\&\subseteq\Pi(1,2,\dots,i)\Pi(i+1,i+2,\dots,n)\mathcal{G}_i\quad[\text{by }(5.12)]\\&\sim\mathcal{G}_i,\end{aligned}$$

which implies the relation (5.14). □

5.5. **Why do we need to pass through something like the permadeath linked Grassmannian in order to obtain the correct linked diagram?** The procedure above is simple and general, but we feel as if an example would clarify why we have to perform this seemingly convoluted procedure in order to obtain the correct doubled quiver with relations (the π-linked chain) to which we attach a linked Grassmannian of the appropriate rank.

Example 5.15. Let's consider the case of $G = GL_3$ with Iwahori level structure (i.e. $I=\{0,1,2\}$), so we are concerned with the local model $M^{loc}=M^{loc}(G,\mathcal{P},\mu)$ attached to

$$(G,\mathcal{P},\mu)=(GL_3,I=\{0,1,2\},\mu=(1,0,0)).$$

The corresponding bound quiver $C_\bullet$ is the cyclic link diagram of length 3 where a full cycle corresponds to multiplication by π. In other words, we have

$$\begin{aligned}C_1&=\mathcal{O}\{e_1,e_2,e_3\}\\C_2&=\mathcal{O}\{\pi e_1,e_2,e_3\}\\C_3&=\mathcal{O}\{\pi e_1,\pi e_2,e_3\},\end{aligned}$$

where the transition maps are given by

$$\begin{tikzcd} C_1 \arrow[r, "\begin{bmatrix} \pi & & \\ & 1 & \\ & & 1 \end{bmatrix}"] & C_2 \arrow[r, "\begin{bmatrix} 1 & & \\ & \pi & \\ & & 1 \end{bmatrix}"] & C_3 \arrow[ll, bend left, "\begin{bmatrix} 1 & & \\ & 1 & \\ & & \pi \end{bmatrix}"] \end{tikzcd}$$

To turn $C_\bullet$ into a doubled quiver that looks like a π-linked chain, we can naively try to construct maps $f^i : C_i \to C_{i-1}$ for $i = 1, 2$ by taking compositions of the maps above. By doing this, we obtain the following chain:

$$\begin{tikzcd} C_1 \arrow[r, bend left, "\begin{bmatrix} \pi & & \\ & 1 & \\ & & 1 \end{bmatrix}"] & C_2 \arrow[l, bend left, "\begin{bmatrix} 1 & & \\ & \pi & \\ & & \pi \end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix} 1 & & \\ & \pi & \\ & & 1 \end{bmatrix}"] & C_3 \arrow[l, bend left, "\begin{bmatrix} \pi & & \\ & 1 & \\ & & \pi \end{bmatrix}"] \end{tikzcd}$$

However, this is not a π-linked chain, because on the special fiber where $\pi = 0$, we have

$$\mathrm{Im}(f^2) \cap \mathrm{Ker}(f^1) = \mathcal{O}\{e_2\} \cap \mathcal{O}\{e_2, e_3\} = \mathcal{O}\{e_2\} \neq 0,$$

which violates condition (III) of (3.1). Thus, we must do something different in order to end up with a π-linked chain.

Thus, we consider the permadeath linked chain $D_\bullet = D_\bullet(C_\bullet)$ associated with the cyclic link diagram $C_\bullet$ is given by

$$\begin{tikzcd} D_1 \arrow[r, bend left, "\begin{bmatrix} \pi & & \\ & 1 & \\ & & 1 \end{bmatrix}"] & D_2 \arrow[l, bend left, "\begin{bmatrix} 1 & & \\ & \pi & \\ & & \pi \end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix} \pi & & \\ & \pi & \\ & & 1 \end{bmatrix}"] & D_3 \arrow[l, bend left, "\begin{bmatrix} 1 & & \\ & 1 & \\ & & \pi \end{bmatrix}"] \end{tikzcd}$$

which is still not a π-linked chain because it violates condition (III) of (3.1): the image of the top left map intersects the kernel of the top right map nontrivially, namely, on $\mathcal{O}\{e_2\}$. But by "exchanging" the upper and lower maps of the permadeath link diagram, we obtain the chain $E_\bullet$ given by

$$\begin{tikzcd} E_1 \arrow[r, bend left, "\begin{bmatrix} 1 & & \\ & \pi & \\ & & \pi \end{bmatrix}"] & E_2 \arrow[l, bend left, "\begin{bmatrix} \pi & & \\ & 1 & \\ & & 1 \end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix} 1 & & \\ & 1 & \\ & & \pi \end{bmatrix}"] & E_3 \arrow[l, bend left, "\begin{bmatrix} \pi & & \\ & \pi & \\ & & 1 \end{bmatrix}"] \end{tikzcd}$$

which is indeed a π-linked chain. $\diamond$

Remark 2. On the special fiber of the example above, the ranks of the maps f_1 and f_2 of $E_\bullet$ are increasing, and moreover are of rank $i_1 - i_0 = 1 - 0 = 1$ and $i_2 - i_0 = 2 - 0 = 2$, respectively. This "rank-increasing" property is an easy consequence of the maps defining a π-linked chain, and it is seen to be a result of the successive compositions that occur in the construction.

6. Application: Analyzing the geometry of linked Grassmannians

We apply the fact that certain linked Grassmannians are unramified local models and use this to study the geometry of the special fiber. We are currently working on trying to apply this knowledge about local models to analyze some problems in the theory of limit linear series [Hwa18], but here we give some previews of how the machinery of local models can be used to understand linked Grassmannians.

Linked Grassmannians attached to link diagram consisting of a single point are simply Grassmannians, so let's consider the next non-trivial class of cases, attached to a π-linked chain of length 2. We saw in Example 1.1 that if the associated dimension vector is $(2, 2)$, and we take the rank 1 linked Grassmannian, then we recover local model of the modular curve $X_0(p)$. It turns out that the case of a general dimension vector (d, d) and rank r is not much harder.

Example 6.1. (Maximal parahoric case) By our theorem, this is precisely the linked Grassmannian studied in [Oss06, Example A.17]. Let $E_\bullet$ be a π-linked chain of length 2 and rank d. Writing $d_1 = \operatorname{rk} f_1$ and $d^1 = \operatorname{rk} f^1$ for the ranks of the maps in the chain, it turns out in the associated linked Grassmannian $\mathrm{LinGr}(r, E_\bullet)$ for $r < d$, there are

$$\ell = \min\{r + 1, d - r + 1, d_1 + 1, d_2 + 1\}$$

irreducible components, each of which are $r(d - r)$-dimensional, and the components are indexed by the dimension of $f_1(V_1)$ on general points. This is agrees precisely with the description given by picking out the elements of the Iwahori orbits parameterizing the μ-admissible set for GL_d, where $\mu = (1^{(n-r)}, 0^r)$. This is even easier to see if you use the μ-permissible set, via the equality given by Kottwitz–Rapoport [KR00]

6.1. A Global Geometric Realization of the Local Model.

One interesting corollary of our theorem is that it shows that global realizations of these local models have already been constructed and studied in literature on limit linear series. One particular instance occurs for what we call Esteves–Osserman (EO) varieties, as they were initially studied by Esteves and Osserman in their work relating limit linear series and fibers of Abel maps in the case of two curves joined at a single node [EO13]. An alternative approach to studying such varieties in greater generality was developed by one of us [Li13]. Here, we present a simplified, concrete approach that complements the ones mentioned above.

Our goal in this section is to apply the following consequence of our main theorem (Theorem 5.1).

Corollary 6.2. *Let $\mathcal{O}$ be a discrete valuation ring. If M^{loc} is a local model attached to the triple $(G, \{\mu\}, \mathcal{L})$ of unramified type A whose associated minuscule cocharacter μ is assumed to be of Drinfeld type, then M^{loc} is isomorphic to an Esteves–Osserman varierty.*

The notion "of Drinfeld type" means that the rank r of the relevant linked Grassmannian is either of dimension 1 or codimension 1. In this setting, explicit equations for this particular local model were also studied by Faltings [Fal97, §4] and Görtz [Gör01] [Gör04]. Both approaches yield different realizations of the same ideals on the relevant local coordinates, but a fundamental difficulty that arises in following these approaches is that the relevant ideals are constructed using equations that come from matrix relations, and it is very tricky to use elementary methods to prove certain geometric properties (e.g. show that varieties are reduced, that is, that the ideals generated by such relations are radical) without making a number of strong assumptions or limiting ourselves to low-rank settings.

However, by realizing this local model as an EO variety leads to quick proofs of many geometric properties of the local model that usually require much more machinery to be developed (e.g. a characterization in terms of affine Weyl groups, notions of μ-admissibility or μ-permissibility, etc.) and gives what seems to be the first cases in the literature of a *global* (as in not just gluing coordinate patches) geometric realization of a local model of a Shimura variety which is not that of the modular curve $X_0(p)$ over $\mathbf{Z}_p$.

The basic results for what we call an Esteves–Osserman variety is developed in [EO13, §4], but it uses a language that is very different from that used in this paper, so we give a more elementary definition of such a variety, at the cost of possibly obscuring some of the connections to notions from the theory of limit linear series on curves.

Definition 6.3. An **Esteves–Osserman (EO) linked graph** $V^\bullet$ is a chain of rank m vector bundles

$$\begin{tikzcd} V^1 \arrow[r, bend left] & V^2 \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & V^d \arrow[l, bend left] \end{tikzcd}$$

where each edge $e = (s(e), t(e)) \in E_{V^\bullet}$ is labeled with the additional data of an integer $r_e \in (0, m) \cap \mathbf{Z}_+$.

Let S be an arbitrary scheme.

Definition 6.4. An **Esteves–Osserman (EO) linked Grassmannian** is the S-scheme representing the generalized linked Grassmannian attached to an EO link graph, where the induced maps $f_e : V^{s(e)} \to V^{t(e)}$ on each tuple of subbundle must have rank r_e.

Definition 6.5. An **Esteves–Osserman (EO) variety** is a subvariety obtained as the image of the embedding of the EO Linked Grassmannian associatd with an EO linked graph $V^\bullet$ into $\mathbf{P}(V^1) \times \mathbf{P}(V^d)$ by mapping each element of a point $F_\bullet$ of the Linked Grassmannian by using the map obtained by following the shortest path to the **extremal** vertices V^1 and V^d and taking the Zariski closure of the union of the induced rational maps.

Example 6.6. Consider the EO variety attached to the chain

$$\begin{tikzcd} V^1 \arrow[r, bend left, "1"] & V^2 \arrow[l, bend left, "2"] \arrow[r, bend left, "2"] & V^3 \arrow[l, bend left, "1"] \end{tikzcd}$$

and label the maps f_1, f^1, f_2, f^2 as we usually do for an s-linked chain, where the numbers above denote the ranks of the maps (e.g. $f_1 : V^1 \to V^2$ is a rank 1 map). For each V^i, there exist a unique shortest composition of maps that gives a map $g_i^1 : V^i \to V^1$ and $g_i^3 : V^i \to V^3$. For example, $f_1^1 = \mathrm{id}_{V_1}$ and $g_1^2 = f_2 \circ f_1$. These g_i^j's are only rational maps, since they are not defined on the kernels.

We will consider the images of these maps. The map

$$g_1 := (g_1^1, g_1^3) : \mathbf{P}(V^1) \dashrightarrow \mathbf{P}(V^1) \times \mathbf{P}(V^3)$$

has image $\mathbf{P}^2 \times \{pt\}$. The map

$$g_2 := (g_2^1, g_2^3) : \mathbf{P}(V^2) \dashrightarrow \mathbf{P}(V^1) \times \mathbf{P}(V^3)$$

has image $\mathbf{P}^1 \times \mathbf{P}^1$. The map

$$g_3 := (g_3^1, g_3^3) : \mathbf{P}(V^3) \dashrightarrow \mathbf{P}(V^1) \times \mathbf{P}(V^3)$$

has image $\{pt\} \times \mathbf{P}^2$.

The closure of the images of these maps

$$X = \overline{(\mathrm{Im}(g_1) \cup \mathrm{Im}(g_2) \cup \mathrm{Im}(g_3))} \subset \mathbf{P}(V^1) \times \mathbf{P}(V^3)$$

is an example of EO variety. This X also admits a description as a Mustafin degeneration of $\mathbf{P}^2$ (i.e. a Mustafin variety, cf. [CHSW11], related to questions of p-adic uniformization and aspects of tropical geometry), which has three components C_1, C_2, C_3 corresponding to the images of the three maps. They all intersect each other; indeed, we have

$$C_i \cap C_j = \mathrm{Im}(g_i) \cap \mathrm{Im}(g_j).$$

By Theorem 6.2 and matching the data to that of the local model, we see that this is nothing other than the local model of attached to GL_3, a cocharcter μ of Drinfeld type, and Iwahori level structure. Thus, this gives us a concrete *global* realization of this local model. $\diamond$

6.2. **EO Varieties are classical linked Grassmannians.** We prove Corollary 6.2 by proving the following result, which seems to be new, although Osserman has informed us that the structure is somewhat implicit in the work [EO13].

Theorem 6.7. *An Esteves–Osserman (EO) variety is a linked Grassmannian attached to an s-linked chain.*

Before we prove this result, we develop some definitions. Let $(v_1, \ldots, v_d)$ represent a point in the EO variety $X = \overline{\bigcup_i \mathrm{Im}(g_i)}$. The proofs of the following almost trivial four results just involve unwinding the definitions using the notions defined above.

Proposition 6.8. *There exists a smallest integer* k *(**the left index**) such that* $f_k(v_k) = v_{k+1}$.

Proof. This follows from conditions (II) and (III) of an s-linked chain (3.1). □

Proposition 6.9. *There exists a largest integer* j *(**the right index**) such that* $f_j = f^j(v_{j+1})$.

Proof. Just like the proof of the previous proposition, this follows from conditions (II) of (III) of an s-linked chain. □

Proposition 6.10. *For exact points, the left index coincides with the right index.*

Corollary 6.11. *Let* $\vec{v} = (v_1, \ldots, v_d)$ *represent a point in the Linked Grassmannian with left index* k *and embed* $\mathrm{LinGr} \hookrightarrow \boldsymbol{P}(V^1) \times \boldsymbol{P}(V^d)$. *Then* $\vec{v}$ *lies in the image of* $g_k : \boldsymbol{P}(V^k) \to \boldsymbol{P}(V^1) \times \boldsymbol{P}(V^d)$.

Proposition 6.12. *For non-exact points, the left index is one less than the right index.*

Corollary 6.13. *Let* $\vec{v} = (v_1, \ldots, v_d)$ *be a point in* LinGr *with right index* k*, and embed* $\mathrm{LinGr} \hookrightarrow \boldsymbol{P}(V^1) \times \boldsymbol{P}(V^d)$. *Then* $\vec{v}$ *lies in the image of*

$$g_k : \boldsymbol{P}(V^k) \to \boldsymbol{P}(V^1) \times \boldsymbol{P}(V^d)$$

and

$$g_{k-1} : \boldsymbol{P}(V^{k-1}) \to \boldsymbol{P}(V^1) \times \boldsymbol{P}(V^d).$$

We can now use these corollaries to prove our main theorem.

Proof of Theorem 6.7. Since the exact points of a Linked Grassmannian are open and dense, the Linked Grassmannian and the EO variety agree on dense opens and are closures in the same ambient space $\mathbf{P}(V^1) \times \mathbf{P}(V^d)$. Thus, they yield isomorphic varieties. □

Since the construction of the EO variety is very concrete, we can even give a description of the global geometry of the local model without even needing to make making use of machinery used to study the special fiber, such as the μ-admissible set in the affine Weyl group.

Example 6.14. Consider the example 6.6 given previously. This is actually the local model of a PEL Shimura variety of type A_2 with Iwahori level structure and minuscule cocharacter of Drinfeld type. There are a number of properties known about this local model, but we will show how many of them can be obtained by using the basic techniques developed to study EO varieties.

There are three irreducible components C_1, C_2, C_3, which come from the closures of the rational maps $g_i : \mathbf{P}(V^i) \to \mathbf{P}(V^1) \times \mathbf{P}(V^3)$ for $i = 1, 2, 3$.

Around the singularity that occurs from the two copies of $\mathbf{P}^2$ touching, the realization of the equations for the local model (or the image of the rational map) shows that its local ring is given by

$$\mathcal{O}[x, y, z]/(xyz - \pi).$$

In particular, this shows that the local model has semistable reduction. In particular, it shows that the non-codimension 1 singularity that occurs from the meeting of $\mathbf{P}^2$'s is transformed into a Cohen–Macaulay singularity by inserting a copy of $\mathbf{P}^1 \times \mathbf{P}^1$ that runs through the intersection of the two copies of $\mathbf{P}^2$.

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