Exponential growth of $H^{4g-6}(\mathcal{M}_g; \mathbf{Q})$, Part I: Universal enveloping algebras and the Poincaré–Birkhoff–Witt theorem

Brian Hwang* (with Claudia Yun** & Shiyue Li**)

*Cornell University, **Brown University

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Slides available at:
http://brianhwang.com/slides/2020stagosaur.pdf

Our story thus far

Our goal: [Chan–Galatius–Payne, Thm. 2.7] To weave the previous results together to show that $H^{4g-6}(\mathcal{M}_g; \mathbf{Q})$ grows at least exponentially in g for $g = 3, 5$, and $g \geq 7$.

What's left?

The general idea: To show the desired growth, we use Willwacher's theorem and Brown's theorem, which together show that there is a free Lie algebra Lie(V) that injects into the H^{4g-6}'s.

Our group will focus on showing how this part, coming from this free Lie algebra, grows exponentially in g.

Your guides:

I. Background on Lie algebras, in particular, universal enveloping algebras and the Poincaré–Birkhoff–Witt theorem (Brian)
II. The key calculation: computation of the Poincaré series of a free Lie algebra (Claudia)
III. How everything ties together to prove our main theorem (Shiyue)

Context: Why are Lie algebras relevant here?

Consider the graded **Q**-vector space

$$V = \mathbf{Q}\langle \sigma_{2i+1} \mid i \geq 1 \rangle$$

where σ_{2i+1} lies in degree $2i+1$.

We're interested in the **free Lie algebra** $\mathrm{Lie}(V)$ on V, that is, it is a graded Lie algebra (i.e. **Q**-vector space with binary operation $[,]$) with no relations other than the minimum necessary to make it a Lie algebra:

1. (bilinearity): $[ax+by, z] = a[x,z] + b[y,z]$ and $[z, ax+by] = a[z,x] + b[z,y]$ for all $a, b \in \mathbf{Q}$ and $x, y, z \in \mathrm{Lie}(V)$
2. (skew-symmetry): $[x,x] = 0$ for all $x \in \mathrm{Lie}(V)$
3. (Jacobi identity): for all $x, y, z \in \mathrm{Lie}(V)$

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

Why do we care?

A. *Willwacher's theorem* gives an isomorphism between the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ and the degree 0 part of graph cohomology $H^0(\mathrm{GC})$. *Brown's theorem* says that $\mathfrak{grt}_1$ contains the free Lie algebra $\mathrm{Lie}(V)$ generated by symbols σ_{2i+1} for each $i \geq 1$. Together, we obtain a grading-preserving injection

$$\mathrm{Lie}(V) \hookrightarrow \mathfrak{grt}_1 \cong H^0(\mathrm{GC}) \cong \left(\bigoplus_{g \geq 2} H_0(G^{(g)}) \right)^\vee \subset \bigoplus_{g \geq 2} H^{4g-6}(\mathcal{M}_g; \mathbf{Q}).$$

Upshot: This is how we produce a large number of elements in H^{4g-6} and gives us a lower bound on its size (i.e. its dimension as a vector space over $\mathbf{Q}$).

Q. How do we calculate the dimension of $\mathrm{Lie}(V)$?

A. This has a classical answer using standard Lie-theoretic tools, namely, the universal enveloping algebra and the Poincaré–Birkhoff–Witt theorem.

Why the universal enveloping algebra and the PBW theorem?

Q. How do you construct Lie algebras?

The proto-example of a Lie algebra is an arbitrary associative algebra A together with the bracket given by *commutation*:

$$[x, y] = xy - yx.$$

Then $(A, [,])$ satisfies $[x, x] = 0$ and the Jacobi identity.

Not all Lie algebras arise in this way. For example, we could take a "Lie subring" $B \subset A$, that is, an additive subgroup that is closed under commutation. As B does not need to arise from an *associative* subring, this gives a new class of examples.

The Poincaré–Birkhoff–Witt theorem says that these two operations suffice to realize all Lie algebras: exhibit a suitable associative algebra, change the operation to commutation, pass to a suitable Lie subalgebra.

Universal Enveloping Algebra

The **universal enveloping algebra** $U(L)$ of the free Lie algebra $L = \mathrm{Lie}(V)$ defined on a set V is the free (unital) *associative* algebra generated by V, that is, the ∞-dimensional $\mathbf{Q}$-vector space

$$U(L) = \bigoplus_{\text{words } w \text{ in } V} \mathbf{Q}w$$

where multiplication is by concatenation of words:

$$(av_{i_1}v_{i_2}\cdots v_{i_\ell})\cdot(bv_{j_1}v_{j_2}\cdots v_{j_m}) = abv_{i_1}v_{i_2}\cdots v_{i_\ell}v_{j_1}v_{j_2}\cdots v_{j_m}.$$

This multiplication is also $\mathbf{Q}$-bilinear, but is associative, in contrast to $(\mathrm{Lie}(V), [,])$

Mnemonic. $U(\mathrm{Lie}(V))$ is a "polynomial" $\mathbf{Q}$-algebra where the variables are the elements of V, but where the variables *do not* commute ("$v_iv_j \neq v_jv_i$" for $i \neq j$).

What's so universal about $U(L)$?

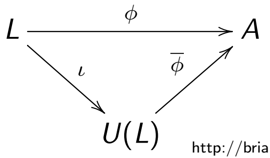
Given a basis $\{x_i\}$ for L, we write $\{X_i\}$ for the corresponding generators of $U(L)$. This gives us a canonical linear map

$$\iota : L \to U(L)$$

defined by $\iota(x_i) = X_i$. In particular, we must have

$$\iota\left(\sum_k a_{ij}(k)x_k\right) = \iota([x_i, x_j]) = X_iX_j - X_jX_i = \sum_k a_{ij}(k)X_k.$$

The algebra $U(L)$ admits the following universal property: *Given any Lie algebra homomorphism $\phi : L \to A$ where A is an associative algebra with its induced (commutator) $[,]$, there exists a unique (algebra) homomorphism $\overline{\phi} : U(L) \to A$ such that the following diagram commutes:*



How does $U(L)$ relate to $L = \text{Lie}(V)$?

This relation is given by the Poincaré–Birkhoff–Witt theorem, which has a number of equivalent formulations and generalizations. Abstractly, it relates an algebra *with a filtration* to its *associated graded* algebra. We use the following formulation.

Thm. (Poincaré–Birkhoff–Witt) If $\{x_1, x_2, \ldots\}$ is a (totally) *ordered* basis for a Lie algebra L, then $\{X_1^{b_1} X_2^{b_2} \cdots \mid b_i \geq 0\}$ is a basis (called a "PBW basis") for the universal enveloping algebra $U(L)$.

Cor. The canonical algebra map $\iota : L \to U(L)$ defined by $\iota(x_i) = X_i$ is *injective*. In particular, L is isomorphic to a Lie subalgebra of an associative algebra.

Rem. We are implicitly using the fact that we are working over a field.

Why is the Poincaré–Birkhoff–Witt theorem a "theorem"?

Q. What makes this result nontrivial?

It's clear that the basis elements $\{X_1^{b_1}X_2^{b_2}\cdots \mid b_i \geq 0\}$ span $U(L)$: we can use the structure constants on $\mathrm{Lie}(V)$

$$[y,z] = \sum_{v\in V} c_{y,z}^v v$$

to reduce an arbitrary product $X_{i_1}^{c_1}X_{i_2}^{c_2}\cdots$ in $U(L)$ to one of the form $X_1^{b_1}X_2^{b_2}\cdots$.

What's not obvious is that these elements are *linearly independent* and that the end result of the reduction is *unique* and is *independent of the order* in which you swap elements to get into "normal form," that is, to lie in the span of our PBW basis.

Example: A case of the PBW Theorem for $L = \mathfrak{sl}_2(\mathbf{C})$

PBW says: *The associative algebra* $U(\mathfrak{sl}_2(\mathbf{C}))$ *has as a vector space basis*

$$\{F^a H^b E^c : a, b, c \geq 0\}.$$

Let's see this in action. Recall that the relations in $U(\mathfrak{sl}_2(\mathbf{C}))$ are

$$EF - FE = H, \quad HE - EH = 2E, \quad HF - FH = -2F.$$

Consider, for example, the following way to get an element into normal form:

$$\begin{aligned} HEF &= H(H + FE) = H^2 + HFE \\ &= H^2 + (FH - 2F)E = H^2 + FHE - 2FE. \end{aligned}$$

It easy to see that this kind of procedure works in general, so the PBW basis at least spans $U(\mathfrak{sl}_2(\mathbf{C}))$.
What's less clear:

- Any other possible sequence of substitution will ultimately result in the same expression in the PBW basis.
- Linear independence of the basis elements (Why are, say, E and F linearly independent?).

Back to proof: What is $U(\mathrm{Lie}(V))$?

Let's return to the free Lie algebra $L = \mathrm{Lie}(V) = \langle \sigma_3, \sigma_5, \sigma_7, \ldots \rangle$ where σ_d lies in degree d. What is $U(L)$?

As L is a free Lie algebra, our construction of $U(L)$ tells us that

$$U(L) \cong \bigoplus_{n \geq 0} V^{\otimes n}.$$

The PBW theorem says that $\{\sigma_3^{d_3} \sigma_5^{d_5} \sigma_7^{d_7} \cdots \mid d_i \geq 0\}$ is a basis for $U(L)$. Together, these tell us that as *graded* vector spaces,

$$U(L) \cong \mathrm{Sym}(L) := \bigoplus_{n \geq 0} L^{\otimes n} / I_n$$

where

$$I_n = \left\langle (x_1 \otimes x_2 \otimes \cdots \otimes x_n - x_{\tau(1)} \otimes x_{\tau(2)} \otimes \cdots \otimes x_{\tau_n} \;\middle|\; \tau \in \mathfrak{S}_n \right\rangle.$$

Summary and towards the key calculation

What have we done so far?

- Defined the Lie algebra $L = \mathrm{Lie}(V)$ contained in $H^6(\mathcal{M}_3; \mathbf{Q}) \oplus H^{14}(\mathcal{M}_5; \mathbf{Q}) \oplus \bigoplus_{g \geq 7} H^{4g-6}(\mathcal{M}_g; \mathbf{Q})$.
- Constructed an associative algebra $U(L)$ that contains L as a Lie subalgebra (with respect to the commutator $[,]$ on $U(L)$).
- Used the Poincaré–Birkhoff–Witt theorem to show that

$$U(\mathrm{Lie}(V)) \cong \mathrm{Sym}(\mathrm{Lie}(V)) \qquad (*)$$

as graded vector spaces.

Next: Use the identification $(*)$ to calculate the dimension of the degree d-part $\dim_{\mathbf{Q}} \mathrm{Lie}(V)_d$ of $\mathrm{Lie}(V)$ and determine the growth rate as $d \to \infty$.